МІНІСТЕРСТВО ОСВІТИ І НАУКИ УКРАЇНИ
НАЦІОНАЛЬНИЙ ТЕХНІЧНИЙ УНІВЕРСИТЕТ
«Харківський політехнічний інститут»

Lyubchenko O. A.
Любченко О. А.

MECHANICS
МЕХАНІКА

Навчально-методичний посібник
з курсу «Загальна фізика»
для студентів усіх спеціальностей та усіх форм навчання

Затверджено
редакційно-видавничою
радою університету,
протокол № 1 від 03.02.2016 г.

Харків
НТУ «ХПІ»
2016

УДК 531+534(07)
ББК 22.3
Л935

Рецензенти:

Є. С. Сиркин, д-р фіз.-мат. наук, провідн. н. с. ФТІНТ НАНУ

А. П. Сук, канд. фіз.-мат. наук, проф. каф. ОЕФ НТУ «ХПИ»

Навчально-методичний посібник з курсу «Загальна фізика» містить теоретичний матеріал, методичні вказівки до розв'язання задач з прикладами розв'язаних задач та контрольні задачі з відповідями з теми «Механіка» (розділи «Кінематика», «Динаміка» та «Коливання та хвилі»).

Призначено для студентів усіх спеціальностей та усіх форм навчання, що навчаються англійською мовою.

Lyubchenko O. A.
Л935 Mechanics. Study Guide / O. A. Lyubchenko – Kh. : NTU "KhPI". 2016. – 324 p. – English Lang.

ISBN

The study guide on "General Physics" contains the theoretical material, solutions manual with examples of solved problems, and set of control questions with answers on "Mechanics" (chapters "Kinematics", "Dynamics" and "Oscillations and Waves").

This study guide is intended for students of all specialties of all modes of study.

Іл. 125. Табл. 9. Бібліогр.: 12 назв.

УДК 531+534(07)
ББК 22.3

ISBN

PREFACE

The goal of this book is to introduce the first year students of National Technical University “Kharkov Polytechnic Institute” with different prior knowledge of physics and incoming language proficiency to the basic concepts and methods of the “Mechanics” as the part of the “General Physics” course.

The book consists of three chapters “Kinematics”, “Dynamics” and “Oscillations and Waves”. Each chapter contains three parts: the study guide, the problem solving guide and the part with control questions.

The study guide presents the theoretical material in form of lecture notes. It includes the basic concepts and definitions, the laws and formulas, the theoretical derivations and practical examples. Text of lectures is supplied by necessary illustrations, figures, diagrams, schemes that enhance the understanding of material.

The second part of each chapter presents the comprehensive overview of the main methods of the problem solution. It is the set of solved problems with detailed and extensive analysis and explanations of the solutions. It covers all aspects of the topics that were under consideration during lectures. Successive study of the theoretical and practical parts allows taking in a lecture material successfully. The complete solutions with explicit calculations and many physical comments make it possible to use this book as the additional help at the lecture course study, and what is more, it can be used by students on their own, at their self-directed individual work. The control questions and problems contained in the third part of each chapter are supplied by answers that help students to estimate the level of retention of auditory and self-studied material.

The structure of the book helps students, who want to familiarize themselves with “Mechanics”, to recognize the material which they should study in addition to the lectures to become acquainted, chapter by chapter, with the entire field of this part of physics.

Chapter 1. KINEMATICS

1. 1. KINEMATICS OF THE TRANSLATIONAL MOTION

1.1.1. The basic concepts and definitions

The general study of the relationships between motion, forces and energy is called ***mechanics***. It is a large field and its study is essential to understanding physics, therefore, these chapters appear first. Mechanics can be divided into subdisciplines by combining and recombining its different aspects. Three of them are given special names. The study of motion without regard to the forces or energies that may be involved is called ***kinematics***. It is the simplest branch of mechanics. In contrast, the study of forces without changes in motion or energy is called ***statics***. Lastly, the branch of mechanics that deals with both motion and forces together is called ***dynamics***.

The models used in mechanics are: a point mass or material point (the size or dimensions of the moving object can be neglected in comparison to distances covered by it), a system of material points, a rigid body, and continuous medium.

Motion happens in ***space*** and in ***time***. According to Newton, «absolute space, in its own nature, without regard to anything external, remains always similar and immovable», and «absolute, true and mathematical time, of itself, and from its own nature flows equably without regard to anything external, and by another name is called duration». Newton believed that time and space were separate and that time was absolute wherever in the universe it was measured. Well after, Einstein in his Theory of Relativity suggested that time and space are connected inseparably and that they form four-dimensional “space-time”, which can be sped up or slowed down by energy and matter.

There are several *types of motion*:

1. ***Translational*** motion results in a change of location (any straight line rigidly fastened with a body moves in a parallel way to itself);

2. ***Rotational*** motion occurs when an object spins (any straight line rigidly fastened with a body turns by an angle);

3. ***Oscillatory*** motion is repetitive and fluctuates between two locations;

4. ***Chaotic*** motion is predictable in theory but unpredictable in practice, which makes it appear random.

To describe the motion it is necessary to introduce a *frame of reference* consisting of a coordinate system and a clock.

1.1.2. Actual path, distance, displacement

An ***actual path*** (or *trajectory*) is a line (fig. 1.1) which is described by a material point or an object during its three-dimensional motion. According to the form of a path the motion may be ***rectilinear*** or ***curvilinear*** motion. The special case of curvilinear motion is the rotation.

A ***distance*** is a scalar measure of interval between two locations measured along the actual path connecting them. The symbol for distance is Δs. The origin of this symbol is from the Latin word *spatium.* The distance is measured in meters, expresses by positive number and added arithmetically.

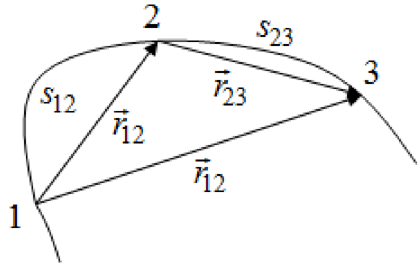


Figure 1.1

$[s]$ = meter = m,

$$s_{13} = s_{12} + s_{23}. \tag{1.1}$$

A ***displacement*** is a vector measure of interval between two locations measured along the shortest path connecting them. $\Delta\vec{r}$ is the symbol for displacement. The origin of this symbol is from the Latin word *radius.* As a vector, displacement has magnitude, direction and a point of origin. A displacement is always a straight line segment from one point to another point, even though the actual path of the moving object between the two points is curved. Furthermore, displacements are vector quantities and can be combined like other vector quantities.

$$\vec{r}_{13} = \vec{r}_{12} + \vec{r}_{23}. \tag{1.2}$$

The magnitude of displacement approaches distance as distance approaches zero.

The SI unit for displacement is the meter.

$[r]$ = meter = m.

To specify the position of an object the concept of the position vector is to be introduced. The *position vector* $\vec{r}$ is defined as a vector that starts at the (user defined) origin and ends at the current position of the object. In general, the position vector $\vec{r}$ will be time dependent $\vec{r}(t)$.

$$\vec{r}(t) = x(t)\cdot\vec{e}_x + y(t)\cdot\vec{e}_y + z(t)\cdot\vec{e}_z, \tag{1.3}$$

where $\vec{e}_x$, $\vec{e}_y$, $\vec{e}_z$ are the unit vectors of the Cartesian frame.

1.1.3. Speed and velocity

Speed is the rate of change of distance with time.

$$v = \frac{s}{t}. \tag{1.4}$$

If a particle moving in a straight line travels equal distances in equal intervals of time, no matter how small these distances may be, the particle is said to be moving with *uniform speed*.

$$v = \frac{\Delta s}{\Delta t} = \frac{s}{t}. \tag{1.5}$$

In case of *non-uniform motion* the *mean* (or *average*) *speed* is introduced as total distance divided by total time, or, the distance traveled along a path divided by the time it takes to travel this distance.

$$\langle v \rangle = \frac{s}{t} = \frac{\text{total distance travelled}}{\text{total time taken}}. \tag{1.6}$$

Car's speedometer shows its instantaneous speed, that is, the speed determined over a very small interval of time – an instant. Ideally this interval should be as close to zero as possible, but in reality we are limited by the sensitivity of our measuring devices. Mentally, however, it is possible to imagine calculating average speed over ever smaller time intervals until we have effectively calculated *instantaneous speed*. This idea is written symbolically as

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}. \tag{1.7}$$

Speed is the first derivative of distance with respect to time.

Both instantaneous speed and average speed are completely described in terms of magnitude alone. Hence *speed* is *scalar*.

When both speed and direction are specified for the motion, the term velocity is used. ***Velocity*** is the rate of change of displacement with time. If, at any point of travel, $\Delta\vec{r}$ is the small change in displacement in a small time interval Δt, the velocity is given by

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} = \dot{\vec{r}}\,. \tag{1.8}$$

The *instantaneous velocity* is a velocity at given instant of time; it is the first derivative of displacement with respect to time. Instantaneous velocity is a vector tangent to the path of moving point (object) (fig. 1.2).

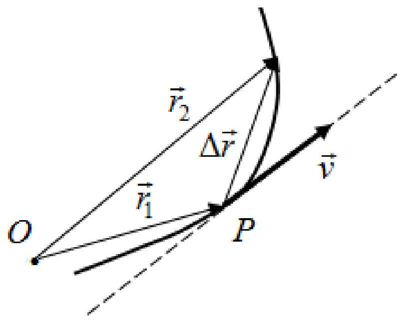


Figure 1.2

Speed and velocity are related in much the same way as distance and displacement are related. Displacement is measured along the shortest path between two points and thus its magnitude is always less than or equal to the distance. The magnitude of the displacement approaches the distance as distance approaches zero. That is, distance and displacement are effectively the same (have the same magnitude) when the interval examined is "small". Since speed is based on distance and velocity is based on displacement, these two quantities are effectively the same (have the same magnitude) when the time interval examined is "small" or, in the language of calculus:

$$v = |\vec{v}| = \left|\lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t}\right| = \lim_{\Delta t \to 0} \frac{|\Delta\vec{r}|}{\Delta t} = \lim_{\Delta t \to 0} \frac{|\Delta\vec{r}|}{\Delta t} \cdot \frac{\Delta s}{\Delta s} = \lim_{\Delta t \to 0} \frac{|\Delta\vec{r}|}{\Delta s} \cdot \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}. \tag{1.9}$$

The magnitude of an object's velocity approaches its speed as the time interval approaches zero.

The velocity vector can be resolved into its three components. The components of velocity can be calculated using the components of position vector as

$$\vec{v} = \dot{\vec{r}} = v_x \cdot \vec{e}_x + v_y \cdot \vec{e}_y + v_z \cdot \vec{e}_z = \dot{x} \cdot \vec{e}_x + \dot{y} \cdot \vec{e}_y + \dot{z} \cdot \vec{e}_z = v \cdot \vec{e}_v, \tag{1.10}$$

where v_x, v_y, v_z are the components of velocity, and $\vec{e}_x$, $\vec{e}_y$, $\vec{e}_z$ are unit vectors of the Cartesian frame.

The SI unit of speed and velocity is the meter per second.

$[v]$ = meter/second = m/s = m·s^{-1}.

1.1.4. Acceleration

The average acceleration of a particle as it moves from one position to another is defined as the change in the instantaneous velocity vector $\Delta\vec{v}$ divided by the time Δt during which that change occurred

$$\vec{a} = \frac{\vec{v}_2 - \vec{v}_1}{\Delta t} = \frac{\Delta\vec{v}}{\Delta t}. \tag{1.11}$$

The ***instantaneous acceleration*** is defined as the limiting value of the change in velocity with time as Δt approaches zero

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \dot{\vec{v}} = \ddot{\vec{r}}. \tag{1.12}$$

Acceleration is the first derivative of a velocity with respect to time and the second derivative of a displacement with respect to time.

The SI unit of acceleration is the meter per second squared.

$[a]$ = m·s^{-2}.

Any change in the velocity of an object results in an acceleration: increasing speed (what people usually mean when they say acceleration), decreasing speed (also called deceleration or retardation), or changing direction. Really, a change in the direction of motion results in an acceleration even if the speed doesn't change. That's because acceleration depends on a change in velocity, and velocity is a vector quantity – one with both magnitude and direction.

Taking into account that $\vec{v} = v \cdot \vec{e}_v$, we‘ll consider three types of motions.

1. The magnitude of velocity (the speed) is changing during straight-line motion

$v \neq$ const; $\vec{e}_v =$ const (*non-uniform rectilinear motion*)

$$\vec{a} = \dot{\vec{v}} = \frac{d}{dt}(v \cdot e_v) = \frac{dv}{dt} \cdot \vec{e}_v = \dot{v} \cdot \vec{e}_v = \vec{a}_\tau. \tag{1.13}$$

The ***tangential*** acceleration $\vec{a}_\tau$ is the component of acceleration vector tangent to the curvilinear path. It characterizes the change of velocity magnitude (speed).

2. The direction of velocity vector is changing with time while speed remains constant

$v = \text{const}; \quad \vec{e}_v \neq \text{const}$ (*uniform curvilinear motion*)

$$\vec{a} = \dot{\vec{v}} = \frac{d}{dt}(v \cdot e_v) = v \cdot \frac{d\vec{e}_v}{dt} = v \cdot \dot{\vec{e}}_v . \tag{1.14}$$

Let's find out what $\dot{\vec{e}}_v$ is. According to definition of a derivative

$$\dot{\vec{e}}_v = \frac{d\vec{e}_v}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \vec{e}_v}{\Delta t} . \tag{1.15}$$

During the small period of time Δt the unit vector of a velocity $\vec{e}_v$ (fig. 1.3, *a*) turns through an angle of

$$\Delta\varphi = \frac{\Delta s}{R} = \frac{v \cdot \Delta t}{R} . \tag{1.16}$$

When $\Delta t \to 0$ and $\Delta\varphi \to 0$, the chord $AB \cong \overset{\frown}{AB}$ (the arc length). Then, taking into account (1.16), $\Delta\vec{e}_v \approx \Delta\varphi \cdot \vec{n}'$. When Δt tends to zero, $\vec{n}' \to \vec{n}$ (fig. 1.3, *b*).

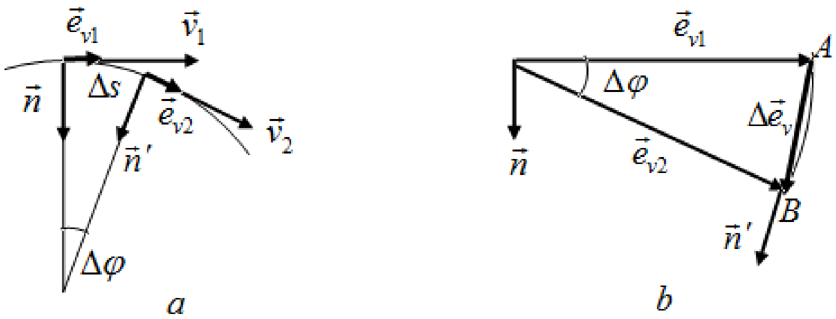


Figure 1.3

$$\dot{\vec{e}}_v = \frac{d\vec{e}_v}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\vec{e}_v}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta\varphi \cdot \vec{n}'}{\Delta t} = \lim_{\Delta t \to 0} \frac{v \cdot \Delta t \cdot \vec{n}'}{R \cdot \Delta t} = \frac{v}{R} \cdot \vec{n} . \tag{1.17}$$

Therefore,

$$\vec{a} = \dot{\vec{v}} = \frac{d}{dt}(v \cdot \vec{e}_v) = v \cdot \frac{d\vec{e}_v}{dt} = v \cdot \dot{\vec{e}}_v = \frac{v^2}{R} \cdot \vec{n} = \vec{a}_n . \tag{1.18}$$

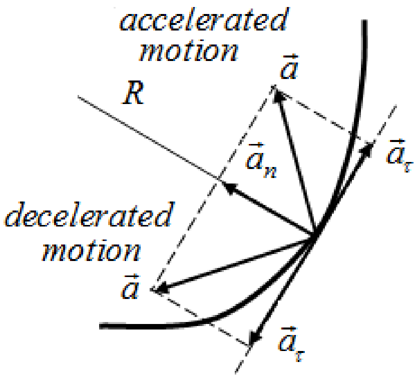


Figure 1.4

The ***normal*** (or *centripetal*, or *radial*) acceleration $\vec{a}_n$ is the component of the acceleration along the in-out direction, i. e., parallel to the radius. It characterizes the change of velocity direction.

3. Both magnitude and the direction of the velocity vector are changing, $v \neq \text{const}$; $\vec{e}_v \neq \text{const}$ (*non-uniform curvilinear motion*). Then, considering (1.14) and (1.18), the acceleration (fig. 1.4) is

$$\vec{a} = \dot{\vec{v}} = \frac{d}{dt}(v \cdot e_v) = \frac{dv}{dt} \cdot \vec{e}_v + v \cdot \frac{d\vec{e}_v}{dt} = \vec{a}_\tau + \vec{a}_n, \qquad (1.19)$$

and its magnitude is

$$a = |\vec{a}| = \sqrt{a_\tau^2 + a_n^2}\,. \qquad (1.20)$$

1.1.5. Relativity of motion. Addition of velocities. Galileo's transformation

If the motion of a particle is considered relative to two frames of reference, K and K', whose respective axes remain parallel to each other, and reference frame K' is moving respectively the reference frame K with velocity $\vec{V}$ along x-axis (fig. 1.5).

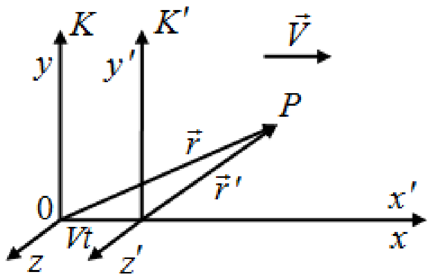


Figure 1.5

Coordinates of the point P and time in these frames of reference are

$$\begin{cases} x' = x + Vt, \\ y' = y, \\ z' = z, \\ t' = t. \end{cases} \qquad (1.21)$$

These equations are called the ***Galileo's transformation*** for coordinates and time.

The position of the point P is determined by radius-vectors $\vec{r}$ and $\vec{r}'$ in reference frames K and K', respectively, and they are related as

$$\vec{r}' = \vec{r} + \vec{V}t\,. \tag{1.22}$$

Using the fact that the time is the same in both reference frames, we can differentiate this equation with time and obtain the ***velocity-addition formula***

$$\frac{d\vec{r}'}{dt} = \frac{d\vec{r}}{dt} + \vec{V}\,,$$

$$\vec{v}' = \vec{v} + \vec{V}\,. \tag{1.23}$$

The second differentiation gives the equality of accelerations in all frames of reference uniformly moving respectively each other.

$$\vec{a}' = \vec{a}\,. \tag{1.24}$$

1.2. ROTATIONAL KINEMATICS

1.2.1. The quantities describing the rotation

Rotational motion is the motion during which all points of a body move along the circular paths which centers lie on one straight line called an *axis of rotation*. Note the difference between *circular* motion and *rotational* motion. During circular motion, the axis of the motion is *outside* the object. During rotational motion, the axis of the motion is *inside* the moving object. A spinning wheel is in rotational motion; an object on the rim of the wheel describes circular motion.

An ***angular displacement*** $\vec{\varphi}$ is the angle about the axis of rotation through which the object turns. The magnitude of the angular displacement is equal to the angle swept out by the point. The direction of the vector $\vec{\varphi}$ is determined by *right hand rule*: orient your right hand so that your thumb is perpendicular to the plane of rotation. If you can curl your fingers in the direction of rotation, your thumb will point in the direction of $\vec{\varphi}$.

$[\varphi]$ = radian = rad.

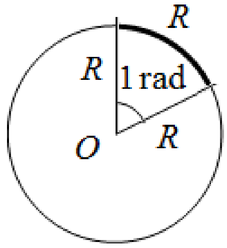


Figure 1.6

The radian is the standard unit of angular measure. 1 radian is the central angle subtended by the arc of circle equaled to the radius of this circle; 1 rad ≈ 57.3° (fig. 1.6). Relation between the angles measured in radians and degrees is

$$\varphi(\text{rad}) = \frac{\pi}{180^\circ} \cdot \varphi(\text{deg}). \qquad (1.25)$$

Another unit for angle measurement is the amount of revolutions N

$$\varphi = 2\pi N. \qquad (1.26)$$

The ***angular speed*** for uniform circular motion and the ***average angular speed*** for non-uniform motion is the ratio of the angle of rotation $\Delta\varphi$ to the time interval Δt taken for this rotation

$$\omega = \frac{\varphi_2 - \varphi_1}{\Delta t} = \frac{\Delta\varphi}{\Delta t}. \qquad (1.27)$$

According to the right hand rule, we take ω to be positive when φ is increasing (counterclockwise motion) and negative when φ is decreasing (clockwise motion) (fig. 1.7).

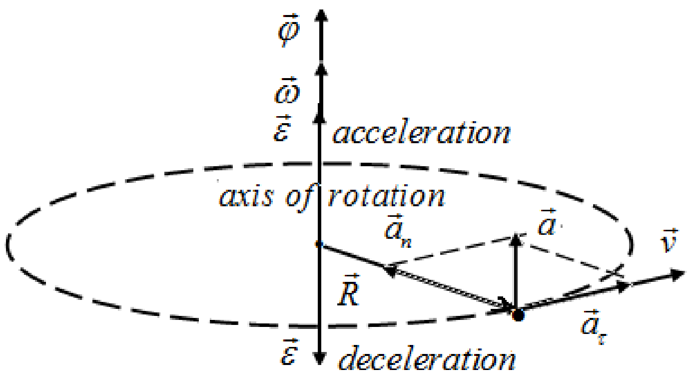


Figure 1.7

For linear motion, a velocity is defined as the time rate of displacement change. Similarly, for rotational motion, the ***instantaneous angular velocity*** is defined as the time rate of angular displacement.

$$\vec{\omega} = \lim_{\Delta t \to 0} \frac{\Delta \vec{\varphi}}{\Delta t} = \frac{d\vec{\varphi}}{dt} = \dot{\vec{\varphi}}. \qquad (1.28)$$

$$[\omega] = \frac{\text{radian}}{\text{second}} = \text{s}^{-1}.$$

The angular speed may be measured in revolutions per second. In that case the term frequency n is used. Relationship between angular speed and frequency is

$$\omega = 2\pi n. \qquad (1.29)$$

If the instantaneous angular speed of an object changes from ω_1 to ω_2 in the time interval Δt, the object has an angular acceleration.

The ***average angular acceleration*** ε of a rotating object is defined as the ratio of the change in the angular speed to the time interval Δt

$$\varepsilon = \frac{\omega_2 - \omega_1}{\Delta t} = \frac{\Delta \omega}{\Delta t}. \qquad (1.30)$$

The ***instantaneous angular acceleration*** is defined as the limit of the ratio $\frac{\Delta \omega}{\Delta t}$ as Δt approaches zero

$$\vec{\varepsilon} = \lim_{\Delta t \to 0} \frac{\Delta \vec{\omega}}{\Delta t} = \frac{d\vec{\omega}}{dt} = \dot{\vec{\omega}}. \qquad (1.31)$$

$$[\varepsilon] = \frac{\text{radian}}{\text{s}^2} = \text{s}^{-2}.$$

Note that ε is positive when the rate of counterclockwise rotation is increasing or when the rate of clockwise rotation is decreasing. When the velocity increases (acceleration), vectors $\vec{\omega}$ and $\vec{\varepsilon}$ coincide in direction; when the velocity decreases (deceleration), they are opposite directed vectors.

Vectors $\vec{\varphi}$, $\vec{\omega}$ and $\vec{\varepsilon}$ are so-called *pseudo-vectors*, i. e., the vectors which direction gets out conditionally.

1.2.2. Relationships between the parameters describing translational and rotational motions

Linear speed v of the point moving at the angular speed ω along the circular path of radius R is

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \varphi \cdot R}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \varphi}{\Delta t} \cdot R = \omega \cdot R . \tag{1.32}$$

Linear velocity related to angular velocity as

$$\vec{v} = \left[\vec{\omega}, \vec{R} \right]. \tag{1.33}$$

Normal or centripetal acceleration a_n connected to linear and angular speeds as

$$a_n = \frac{v^2}{R} = \omega^2 \cdot R . \tag{1.34}$$

Tangential acceleration $\vec{a}_\tau$ related to angular acceleration $\vec{\varepsilon}$ as

$$\vec{a}_\tau = \left[\vec{\varepsilon}, \vec{R} \right]. \tag{1.35}$$

Differentiation of (1.32) gives

$$a_\tau = \dot{v} = \dot{\omega} \cdot R = \varepsilon \cdot R . \tag{1.36}$$

1.2.3. The straight line and rotational motions with constant acceleration

There are several useful equations that can help in solving kinematics problems related to the motion along straight line is with constant acceleration.

The relation between distance s covered during time of the motion t, and speed v after time t of the motion

$$\begin{cases} s = v_o t \pm \dfrac{at^2}{2}, \\ v = v_0 \pm at, \end{cases} \tag{1.37}$$

where v_0 is the initial speed (speed at $t = 0$) and a is the magnitude of the acceleration; signs ‘plus’ and ‘minus’ relate to the accelerated and decelerated motions.

If the acceleration is excluded from the equations (1.37), the distance may be expressed as

$$s = \frac{(v \pm v_0)}{2} t. \tag{1.38}$$

The distance, final and initial speeds and acceleration are related as

$$s = \frac{v^2 - v_0^2}{2a}. \tag{1.39}$$

There are similar equations for the rotational motion with constant angular acceleration ε in terms of initial ω_0 and final ω angular speeds, initial n_0 and final n frequencies, angle of rotation φ and the number of revolutions N of the rotating object.

$$\begin{cases} \varphi = \omega_0 t \pm \dfrac{\varepsilon t^2}{2}, \\ \omega = \omega_0 \pm \varepsilon t, \end{cases} \tag{1.40}$$

$$\begin{cases} 2\pi N = 2\pi n_0 t \pm \dfrac{\varepsilon t^2}{2}, \\ 2\pi n = 2\pi n_0 \pm \varepsilon t. \end{cases} \tag{1.41}$$

$$\varphi = \frac{(\omega \pm \omega_0) t}{2}, \tag{1.42}$$

$$\varphi = \frac{\omega^2 - \omega_0^2}{2\varepsilon}. \tag{1.43}$$

PROBLEMS

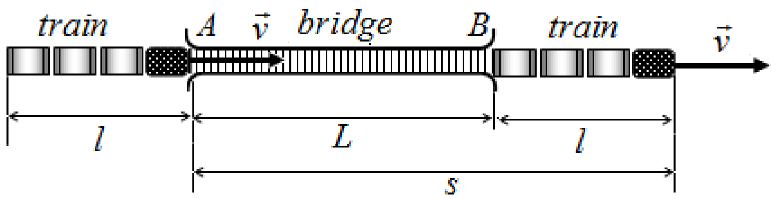
Problem 1.1

A train 100 *m long is moving with a speed of* 60 *km/hr. How long does it take to cross a bridge 1 km long?*

Solution

The train's motion over the bridge begins when the nose of the train pulls into the bridge (point A), and finishes when rear end of the train pulls off the bridge (point B), therefore, the total distance to be covered is

$$s = L + l = 1000 + 100 = 1100 \text{ m}.$$



If the speed of the train is $v = 60$ km/hr $= \dfrac{60000}{3600} = 16.67$ m/s, the sought time is

$$t = \frac{s}{v} = \frac{1100}{16.67} = 66 \text{ s}.$$

Problem 1.2

Find the average speed of the cyclist if he is travelling along a straight road a) for the first half time with speed v_1 *and for the second half time with speed* v_2*; b) for the first half of distance with speed* v_1 *and for the second part of distance with speed* v_2.

Solution

The average speed over any interval of time is defined as the ratio of the distance travelled divided by total time taken.

a) Let t be the total time taken.

Distance covered during the first half time is

$$s_1 = v_1 \cdot \left(\frac{t}{2}\right) = \frac{v_1 \cdot t}{2}.$$

Distance covered during the second half is

$$s_2 = v_2 \cdot \left(\frac{t}{2}\right) = \frac{v_2 \cdot t}{2}.$$

The average speed is

$$v_{ave} = \frac{s_1 + s_2}{t} = \frac{\frac{v_1 \cdot t}{2} + \frac{v_2 \cdot t}{2}}{t} = \frac{t(v_1 + v_2)}{2 \cdot t} = \frac{v_1 + v_2}{2}.$$

b) Let s be the total distance travelled.
Time taken for the first half of distance is

$$t_1 = \frac{(s/2)}{v_1} = \frac{s}{2 \cdot v_1}.$$

Time taken for the second half of distance is

$$t_2 = \frac{(s/2)}{v_2} = \frac{s}{2 \cdot v_2}.$$

The total time taken is

$$t = t_1 + t_2 = \frac{s}{2 \cdot v_1} + \frac{s}{2 \cdot v_2} = \frac{s(v_1 + v_2)}{2 \cdot v_1 \cdot v_2}.$$

The average speed is

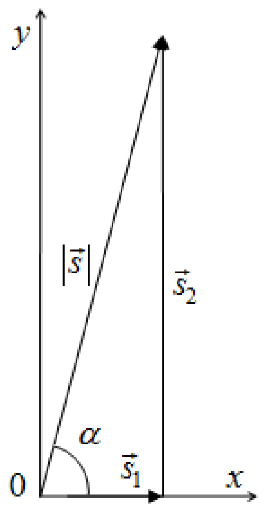
$$v_{ave} = \frac{2 \cdot v_1 \cdot v_2 \cdot s}{s \cdot (v_1 + v_2)} = \frac{2 \cdot v_1 \cdot v_2}{v_1 + v_2}.$$

Problem 1.3

The particle moves for t_1 = 5 *s with the velocity* v_1 = 5 *m/s due east and for* t_2 = 36 *sec with the velocity* v_2 = 4 *m/s due north. Find the distance covered for the time of motion; the displacement of the particle; the average speed; and the average velocity. If the particle moving at the velocity* v_3 = *3 m/s covered the distance to the initial point along the same path, find the average speed and velocity.*

Solution

The distances covered for the first and second periods of motion, respectively, are



$$s_1 = v_1 \cdot t_1 = 5 \cdot 5 = 25 \text{ m},$$

$$s_2 = v_2 \cdot t_2 = 4 \cdot 36 = 144 \text{ m}.$$

The total distance is

$$s = s_1 + s_2 = 25 + 144 = 169 \text{ m}.$$

The average speed which is the distance covered divided by the time taken is given by

$$v_{ave} = \frac{s}{t} = \frac{s_1 + s_2}{t_1 + t_2} = \frac{25 + 144}{5 + 36} = \frac{169}{41} = 4.1 \text{ m/s}.$$

Displacement $\vec{s}$ is the vector connecting the initial and the final points of the motion, therefore, it is $\vec{s} = \vec{s}_1 + \vec{s}_2$.

The magnitude of the displacement may be found using Pythagorean Theorem

$$|\vec{s}| = \sqrt{s_1^2 + s_2^2} = \sqrt{25^2 + 144^2} = \sqrt{21361} = 146.2 \text{ m}.$$

The magnitude of the average velocity is the ratio of displacement and the corresponding time interval, therefore,

$$v_{ave} = \frac{|\vec{s}|}{t} = \frac{146.2}{5 + 36} = 3.57 \text{ m/s}.$$

The time for return path is

$$t_3 = \frac{s_3}{v_3} = \frac{s}{v_3} = \frac{169}{3} = 56.3 \text{ s.}$$

New value of the average speed is

$$v_{ave1} = \frac{2 \cdot s}{t_1 + t_2 + t_3} = \frac{2 \cdot 169}{5 + 36 + 56.3} = 3.47 \text{ m/s.}$$

The average velocity is zero because the displacement in this case is zero (the initial point is the final points of the motion).

Problem 1.4

A boat speed in still water is v_0 =2 *m/s. The river is* 100 *m wide, and the speed of river current is* u =*1 m/s. a) If the boat starts at the south shore and is to reach north shore just opposite the starting point (as shown in Figure), at what angle must the boat head? Find the time for crossing the river. b) If the boat heads directly across the river, find its velocity relative to the shore. How long it take to cross the river and how far downstream will the boat be then?*

Solution

a) If the boat heads straight across the river, the current will drag it downstream (westward). To overcome the river's westward current the boat must acquire an eastward component of velocity as well as northward component. Thus the boat must head in a northeasterly direction. The resultant velocity of the boat

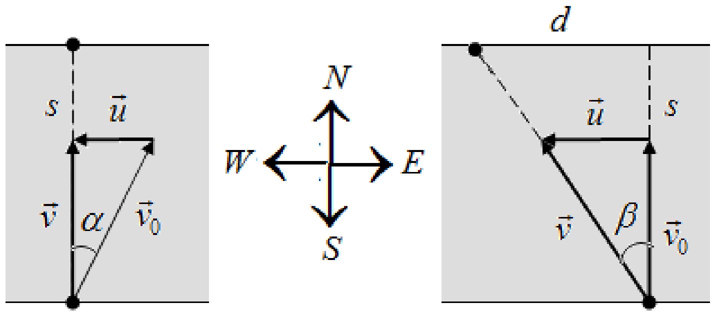
$$\vec{v} = \vec{v}_0 + \vec{u} \, .$$

The magnitude of this velocity is

$$v = \sqrt{v_0^2 - u^2} = \sqrt{2^2 - 1^2} = \sqrt{3} = 1.73 \, \text{m/s.}$$

The vector $\vec{v}$ points upstream at an angle α as shown in Figure. From the diagram,

$$\sin\alpha = \frac{u}{v_0} = \frac{1}{2},$$



Thus, $\alpha = 30^0$, so the boat must head upstream at a 30° angle.

Assuming the uniform motion of the boat, we can find the time for crossing the river

$$t = \frac{s}{v} = \frac{100}{1.73} = 58.8 \text{ s.}$$

b) If the boat heads directly across the river, the current pulls it downstream. The boat's velocity with respect to the shore $\vec{v}$ is the sum of its velocity with respect to water, $\vec{v}_0$, and the velocity of the water with respect to the shore, $\vec{u}$. As a result,

$$\vec{v} = \vec{v}_0 + \vec{u} .$$

Now, using the Pythagorean Theorem, we can find the magnitude of v:

$$v = \sqrt{v_0^2 + u^2} = \sqrt{2^2 + 1^2} = \sqrt{5} = 2.24 \text{ m/s.}$$

The angle we can find from

$$\tan\beta = \frac{u}{v_0} = \frac{1}{2} .$$

Thus, $\beta = 65.9°$. Note that this angle is not equal to the angle α calculated in the first part of the Problem 1.

The time for getting the opposite shore is

$$t = \frac{s}{v} = \frac{100}{2} = 50 \text{ s.}$$

During this time, the boat covers the downstream distance

$$d = u \cdot t = 1 \cdot 50 = 50 \text{ m.}$$

Problem 1.5

A car starts from the rest and accelerates uniformly for t = 5 seconds over a distance of 100 *m. Find the acceleration of the car and the speed at this instant of time.*

Solution

The expressions for the position and speed of an object under constant acceleration are

$$\begin{cases} s = v_0 t + \dfrac{at^2}{2}, \\ v = v_0 + at. \end{cases}$$

Since the car id s starting from rest $v_0 = 0$ and the equations take on form

$$\begin{cases} s = \dfrac{at^2}{2}, \\ v = at. \end{cases}$$

The acceleration may be found from the first equation as

$$a = \frac{2s}{t^2} = \frac{2 \cdot 100}{25} = 8 \text{ m/s}^2.$$

The speed from the second equation is

$$v = at = 8 \cdot 5 = 40 \text{ m/s}.$$

Problem 1.6

A car going at 10 *m/s undergoes an acceleration of* 5 *m/s*^{2} *for 6 seconds. How far did it go when it was accelerating? What is its speed after this period of motion?*

Solution

The kinematic equations that may be used for this problem solution are

$$\begin{cases} s = v_0 t + \dfrac{at^2}{2}, \\ v = v_0 + at. \end{cases}$$

The distance covered for 6 seconds if initial speed is $v_0 = 10$ m/s and acceleration $a = 5$ m/s^2 is

$$s = 10 \cdot 6 + \frac{5 \cdot 6^2}{2} = 150 \text{ m},$$

the speed of the car is

$$v = 10 + 5 \cdot 6 = 40 \text{ m/s}.$$

Problem 1.7

A car accelerates from rest to the speed v_1 =36 *m/s for* 4 *s, and then decelerates to* v_2 = 32 *m/s for next* 6 *s. Find the distance traveled.*

Solution

If the car starts from rest at the uniform acceleration a_1 its speed depends on time as

$$v_1 = v_0 + a_1 t_1 = a_1 t_1 .$$

The acceleration during this motion is

$$a_1 = \frac{v_1}{t_1} = \frac{36}{4} = 9 \text{ m/s}^2.$$

The distance traveled for this time $t_1 = 4$ s is

$$s_1 = \frac{a_1 t_1^2}{2} = \frac{9 \cdot 4^2}{2} = 72 \text{ m}.$$

Acceleration during decelerating motion is equal to

$$a_2 = \frac{v_2 - v_1}{t_2} = \frac{32 - 36}{6} = -\, 0.67 \text{ m/s}^2.$$

The traveled distance is

$$s_2 = v_1 t_2 + \frac{a_2 t_2^2}{2} = 36 \cdot 6 + \frac{(-0.67) \cdot 6^2}{2} = 203.9 \text{ m}.$$

The total distance travelled for the time of motion is

$$s = s_1 + s_2 = 72 + 203.9 = 275.9 \text{ m}.$$

Problem 1.8

If the position of a particle is given by $x = 20t - 5t^3$, where x is in meters and t is in seconds, when if ever is the particle's velocity zero? When is its acceleration a zero? Determine if the motion is accelerated or decelerated.

Solution

To determine velocity, we have to differentiate x with respect to time.

$$v = \frac{dx}{dt} = \frac{d\left(20t - 5t^3\right)}{dt} = 20 - 15 \cdot t^2 .$$

If $v = 0$, then

$$20 - 15 \cdot t^2 = 0, \quad t^2 = 1.33, \quad t = 1.15\,\text{s}.$$

Acceleration is the second-order derivative of position or the first derivative of velocity

$$a = \frac{dv}{dt} = \frac{d\left(20 - 15t^2\right)}{dt} = -30 \cdot t .$$

From this, we see that acceleration can be zero only at $t = 0$; besides that the acceleration is negative whenever t is positive.

Firstly, the motion of the particle is decelerated, because its speed is positive and acceleration is negative. At the instant of time $t = 1.15$ s., the speed of the particle becomes equal to zero. After this, the speed begins to increase in value, while being negative in sign. It means that the particle begins to move in opposite direction. Taking into account the negative sign of acceleration, it can be concluded that this is accelerated motion.

Problem 1.9

The reaction time for an automobile driver is 0.7 *second. If the automobile can be decelerated at* 5 *m/s²$^{}$, calculate the total distance travelled in coming to stop from an initial velocity of* 30 *km/h after a signal is observed.*

Solution

Since the reaction time of the driver is t_1= 0.7 s, therefore, the automobile, during this time, will continue to move with uniform speed of 30 km/h = 8.33 m/s.

The distance covered during 0.7 s is

$$s_1 = v_1 t_1 = 8.33 \cdot 0.7 = 5.83 \text{ m}.$$

Upon that, the automobile begins to decelerate. Relationship between initial velocity $v_1 = 8.33$ m/s and final velocity $v_2 = 0$, acceleration $a = -5\,\text{m/s}^2$, and covered distance s_2 gives

$$s_2 = \frac{v_2^2 - v_1^2}{2a} = \frac{0 - 8.33^2}{-2 \cdot 5} = 6.94 \text{ m}.$$

The total distance travelled is

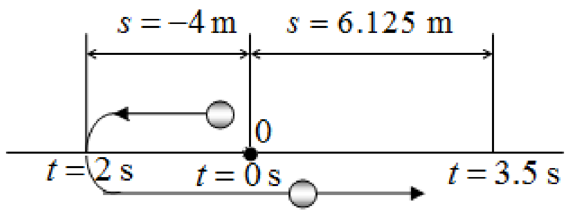
$$s = s_1 + s_2 = 5.83 + 6.94 = 12.77 \text{ m}.$$

Problem 1.10

A particle moves along a horizontal path with a velocity of $v = 3t^2 - 6t$ *(m/s), where* t *is time in seconds. If it is initially located at the origin O, determine the distance travelled in 3.5 s, and the particle's average velocity and average speed during the time interval.*

Solution

Since $v = f(t)$, the position as a function of time may be found by integrating $v = \dfrac{ds}{dt}$. Assuming that at $t = 0$ $s = 0$, we obtain



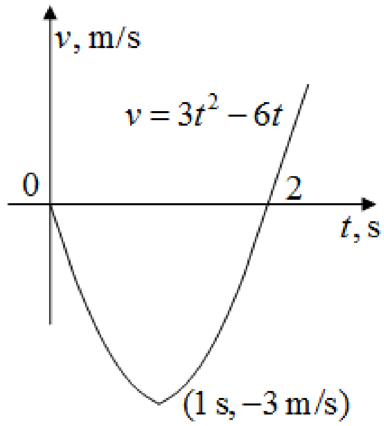
$$ds = vdt = \left(3t^2 - 6t\right)dt,$$

$$\int_0^s ds = \int_0^t \left(3t^2 - 6t\right)dt,$$

$$s = \left(t^3 - 3t^2\right) \text{ m}.$$

In order to determine the distance travelled in 3.5 s, it is necessary to investigate the path of motion. The function $v = 3t^2 - 6t$ is the parabola. From $v = 3t^2 - 6t = 0$, $t = 2$ s, therefore, the graph intersects the x-axis at the point

$t = 2$. To determine the minimum of the function find the magnitude of t when derivative of the function is zero.



$$\frac{dv}{dt} = \left(3t^2 - 6t\right)' = 6t - 6 = 0,$$

$$t = 1 \text{ s}.$$

Thus, for $0 < t < 2$ s the velocity is negative, and the particle is travelling to the left, for $2\text{ s} < t < 3.5\text{ s}$ the velocity is positive, and the particle is travelling to the right. Using $s = \left(t^3 - 3t^2\right)$m we can determine the particle's positions:

$$s\big|_{t=0} = 0, \quad s\big|_{t=2s} = -4 \text{ m}, \quad s\big|_{t=3.5s} = 6.125 \text{ m}.$$

The distance traveled in 3.5 s is

$$s = 4 + 4 + 6.125 = 14.125 \text{ m}.$$

The displacement from $t = 0$ to $t = 3.5$s is

$$\Delta s = s\big|_{t=3.5s} - s\big|_{t=0} = 6.125 - 0 = 6.125 \text{ m}.$$

The average velocity which is equal to the displacement divided by the time of motion is

$$\left|\vec{v}\right|_{ave} = \frac{\Delta s}{\Delta t} = \frac{6.125}{3.5} = 1.75 \text{ m/s}.$$

The average speed is defined in terms of the distance traveled for the time of motion.

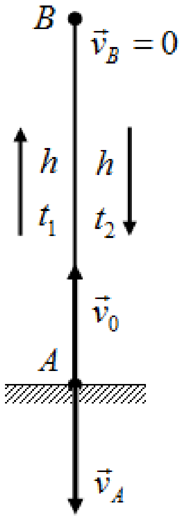
$$v_{ave} = \frac{s}{\Delta t} = \frac{14.1}{3.5} = 4.04 \text{ m/s}.$$

Problem 1.11

A body thrown upwards strikes the ground after $t = 3$ s. Find its initial velocity v_0 and the height h.

Solution

The upwards motion of the body during the time period t_1 is decelerated (uniform acceleration $\vec{g}$ directed downwards meanwhile the initial velocity v_0 directed upwards), and the downward motion during the time period t_2 is accelerated (free fall). The equations describing these motions are



$$\begin{cases} h = v_0 t_1 - \dfrac{g t_1^{\,2}}{2}, \\ v_B = v_0 - g t_1, \\ h = \dfrac{g t_2}{2}, \\ v_A = g t_2, \\ t = t_1 + t_2. \end{cases}$$

As $v_B = 0$ then $v_0 = g t_1$. Substituting v_0 in the first equation of the system we obtain $h = \dfrac{g t_2^2}{2}$. Comparison of this equation with the third equation of the system allows making the conclusion that the time of ascent is equal to the time of descent

$$t_1 = t_2 = \frac{t}{2} = 1.5 \text{ s.}$$

Therefore, the initial velocity is equal to the velocity at hitting the ground

$$v_0 = v_A = g t_1 = 9.8 \cdot 1.5 = 14.7 \text{ m/s.}$$

The height is

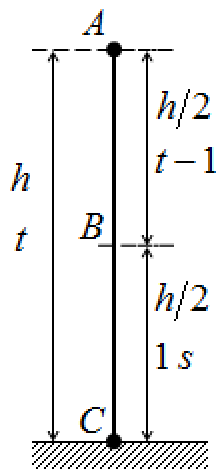
$$h = \frac{g t_1^2}{2} = \frac{9.8 \cdot 1.5^2}{2} = 22.05 \text{ m.}$$

Problem 1.12

Free falling body covered the second half the height during 1 *s. How long does the body take to strike the ground? What was the height?*

Solution

The time dependence of the distance covered by free falling object is $h = \frac{gt^2}{2}$. As the section BC (the half of the whole distance h) is covered for the time period 1 s, then the first half of the distance AC was covered for $(t-1)$ s. Therefore, the motion along the section AB may be described as $\frac{h}{2} = \frac{g(t-1)^2}{2}$. Solving the system



$$\begin{cases} h = \dfrac{gt^2}{2}, \\ \dfrac{h}{2} = \dfrac{g(t-1)^2}{2}, \end{cases}$$

we obtain $t^2 - 4t + 2 = 0$. The roots of this equation are $t_1 = 3.41$ s and $t_2 = 0.59$ s. The second root has to be rejected as the time of motion has to be greater than 1 s. Therefore, the time of free fall is $t_1 = 3,41$ s and the height is

$$h = \frac{gt^2}{2} = \frac{9.8 \cdot 3.41^2}{2} \approx 57 \text{ m}.$$

Problem 1.13

A ball is thrown upward from the top of a 25*-m-tall building. The ball's initial speed is* 12 *m/s. At the same instant, a person is running on the ground at a distance of* 31 *m from the building. What must be the average speed of the person if he is to catch the ball at the bottom of the building?*

Solution

Let's take the origin of the frame of reference at the top of the building. Then the position of the ball at any time is

$$y = v_0 t + \frac{at^2}{2}.$$

Substituting $y = -25$ m, $g = -9.8$ m/s^2, and $v_0 = 12$ m/s in this equation we obtain

$$4.9t^2 - 12t - 25 = 0.$$

The quadratic equation yields solutions $t = 3.79$ s and –1.34 s. The negative solution is rejected as being non-physical. During time t the person has run a distance $s = 31$ m. His speed has to be

$$v = \frac{s}{t} = \frac{31}{3.79} = 8.2 \text{ m/s}.$$

Problem 1.14

A body is thrown up with velocity of 78.4 m/s. Find how high it will rise and how much time it will take to return to its point of projection ($g = 9.8$ m/s^2*).*

Solution

Let use the equations describing accelerated motion

$$\begin{cases} s = v_0 t \pm \dfrac{at^2}{2} \\ v = v_0 \pm at \end{cases}$$

Taking into account that upward motion is decelerated and velocity at the maximum height is zero, rewrite this system as

$$\begin{cases} h = v_0 t - \dfrac{gt^2}{2} \\ 0 = v_0 - gt \end{cases}$$

The time of upward motion from the second equation is

$$t = \frac{v_0}{g} = \frac{78.4}{9.8} = 8 \text{ s}.$$

The height is

$$h = v_0 t - \frac{gt^2}{2} = v_0 \cdot \frac{v_0}{g} - \frac{g}{2}\left(\frac{v_0}{g}\right)^2 = \frac{v_0^2}{2g} = \frac{(78.4)^2}{2 \cdot 9.8} = 313.6 \text{ m}.$$

Downward motion takes

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \cdot 313.6}{9.8}} = 8 \text{ s}.$$

The total time for up and down motion is $t_0 = 2t = 2 \cdot 8 = 16$ s.

Problem 1.15

A ball is dropped from a height of h meters above the ground and at the same instant another ball is projected upwards from the ground. The two balls meet when upper ball falls through a distance h/3. Prove that the velocities of two balls when they meet are in ratio 2:1.

Solution

For the first ball

$$\frac{h}{3} = \frac{gt^2}{2}.$$

For the second ball

$$h - \frac{h}{3} = v_0 t - \frac{gt^2}{2}.$$

Adding these two equations, we obtain

$$\frac{h}{3} + h - \frac{h}{3} = v_0 t - \frac{gt^2}{2} + \frac{gt^2}{2},$$

$$h = v_0 t \text{, and } t = \frac{h}{v_0}.$$

Substitute the obtained time in the first equation

$$\frac{h}{3} = \frac{gt^2}{2} = \frac{g}{2}\left(\frac{h}{v_0}\right)^2.$$

Solve for the initial velocity of the second ball that started from the ground

$$v_0^2 = \frac{g}{2} \cdot 3h,$$

$$v_0 = \sqrt{\frac{3gh}{2}}.$$

The velocity of the second ball at the place where it meets the first ball is

$$v_2^2 - v_0^2 = -2g\left(h - \frac{h}{3}\right),$$

$$v_2^2 = v_0^2 - 2g \cdot \frac{2h}{3},$$

$$v_2^2 = \frac{3gh}{2} - \frac{4gh}{3} = \left(\frac{3}{2} - \frac{4}{3}\right)gh,$$

$$v_2 = \sqrt{\frac{gh}{6}}.$$

Velocity of the first ball at the position where it meets the second ball is given by

$$v_1^2 - 0 = \frac{2gh}{3},$$

$$v_1 = \sqrt{\frac{2gh}{3}}.$$

As a result, the ratio of the velocities of two balls is

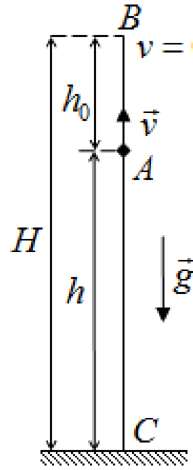
$$\frac{v_1}{v_2} = \sqrt{\frac{2gh \cdot 6}{3 \cdot gh}} = 2.$$

Problem 1.16

A balloon starts rising from the ground with an acceleration of 1.5 *m/s. After* 10 *second, a stone is released from the balloon. Starting from the release of stone, find the displacement and distance traveled by the stone on reaching the ground. Also, find the time taken to reach the ground.*

Solution

Balloon rising at constant acceleration $a = 1.5$ m/s^2 reaches the height h (considering origin on ground and upward direction as positive) for $t = 10$ seconds of the motion.



$$h = \frac{at^2}{2} = \frac{1.5 \cdot 10^2}{2} = 75 \text{ m}.$$

Its velocity at this instant of time is

$$v = at = 1.5 \cdot 10 = 15 \text{ m/s}.$$

The path of motion of the stone is depicted in the figure. The stone released from the balloon (point A) has the velocity $\vec{v}$ directed upwards. The stone is acted upon by the force of gravity alone. As a result, it has upward decelerated motion during the time t_1 with acceleration due to gravity until its velocity is zero.

$$0 = v - gt_1,$$

$$t_1 = \frac{v}{g} = \frac{15}{9.8} = 1.53 \text{ s}.$$

Stone rises to a certain height h_0 above $h = 75$ m where it was released.

$$h_0 = v \cdot t_1 - \frac{gt_1^2}{2} = 15 \cdot 1.53 - \frac{9.8 \cdot 1.53^2}{2} = 11.48 \text{ m}.$$

From this highest vertical point B, the stone falls freely under gravity and hits the ground.

The height of the point B is

$$H = h + h_0 = 75 + 11.48 = 86.48 \text{ m}.$$

The time of the free fall is

$$t_2 = \sqrt{\frac{2H}{g}} = \sqrt{\frac{2 \cdot 86.48}{9.8}} = 4.2 \text{ s}.$$

The total time of the stone motion is

$$t = t_1 + t_2 = 1.53 + 4.2 = 5.73 \text{ s.}$$

The distance travelled by the stone is

$$s = AB + BC = h_0 + H = 11.48 + 86.48 = 97.96 \text{ m.}$$

The magnitude of displacement is the separation between the initial (A) and final (C) points of the motion, therefore, it is equal to

$$AC = h = 75 \text{ m.}$$

Problem 1.17

If a particle's position is given by $x = 6 - 12t + 4t^2$ *(where t is in seconds, and x is in meters), a) what is its velocity at t=*1*s? b) what is its speed at t=*1*s? c) Is there ever an instant when the velocity is 0? If so, give the time.*

Solution

The velocity is the derivative of $x(t)$ with respect to time. Then

$$v(t) = \frac{dx}{dt} = -12 + 8t .$$

a) at t =1 s we get:

$$v\big|_{t=1s} = -12 + 8 = -4 \text{ m/s.}$$

What we calculate here is an x-component of velocity. The negative sign means that the direction of velocity is opposite to the direction of x-axis.

b) the speed is the magnitude of velocity. Then at t = 1 s the speed is 4 m/s

c) to find the instant of time when velocity is zero, we just need to solve the equation:

$$v(t) = -12 + 8t = 0 .$$

From this equation we find time:

$$t = \frac{12}{8} = 1{,}5 \text{ s.}$$

At this instant of time the velocity equal to 0.

Problem 1.18

A rock is dropped from rest into a well. a) The sound of the splash is heard 4 s after the rock is released from rest. How far below to top of the well is the surface of the water? (The speed of sound in air at ambient temperature is 336 m/s). b) If the travel time for the sound is neglected, what % error is introduced when the depth of the well is calculated?

Solution

a) Let h be the depth of the well. To find the time of the rock's fall we use the equation $h = \frac{gt^2}{2}$.

The time of rock's falling down is

$$t = \sqrt{\frac{2h}{g}}.$$

After the rock hits the water the sound of splash propagates the distance h with speed v_s = 336 m/s. The sound reaches the ground level for the time period

$$t_s = \frac{h}{v_s}.$$

The total time is

$$t_0 = t + t_s = \sqrt{\frac{2h}{g}} + \frac{h}{v_s}.$$

This time is equal to t_0 = 4 s. Then

$$4 = \sqrt{\frac{2h}{9.8}} + \frac{h}{336}.$$

$$\left(4 - \frac{h}{336}\right)^2 = \frac{2h}{9.8},$$

$$h^2 - 25728h + 1806336 = 0.$$

This quadratic equation has two roots: one of them $h = 25657$ m is physically meaningless; the second root $h = 70.4$ m gives the desired height.

b) If we neglect the sound velocity then t = 4 s is the time of falling down. The height calculated in this case is

$$h_1 = \frac{gt^2}{2} = \frac{9.8 \cdot 4^2}{2} = 78.4 \text{ m.}$$

If we compare this result with the result of the first part, we can find an error

$$\frac{h_1 - h}{h} \cdot 100\% = \frac{78.4 - 70.4}{70.4} \cdot 100\% = 11.4\%$$

Problem 1.19

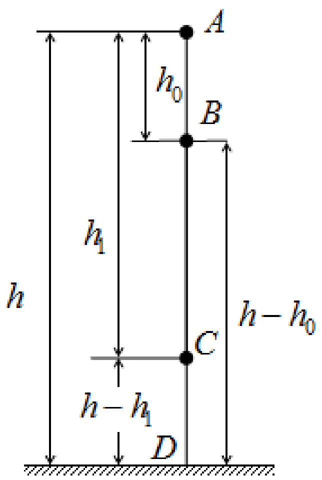
A ball is released from a top. Another ball is dropped from a point $h_1 = 15$ m below the top, when the first ball reaches a point $h_0 = 5$ m below the top. Both balls reach the ground simultaneously. Determine the height h of the top.

Solution

The first ball is free falling from the top and covered the distance $AB = h_0 = \frac{gt_0^2}{2}$ for the time $t_0 = \sqrt{\frac{2h_0}{g}}$. Therefore, the ball's speed at the point B is equal to

$$v_B = gt_0 = g \cdot \sqrt{\frac{2h_0}{g}} = \sqrt{2gh_0} .$$

Now, the first ball covered the distance $BD = h - h_0$ moving with initial velocity $v_B = \sqrt{2gh_0}$ during the time t. For accelerated motion with acceleration due to gravity



$$h - h_0 = v_B t + \frac{gt^2}{2} .$$

The second ball during its free fall covered the distance $CD = h - h_1$ for the same time t

$$h - h_1 = \frac{gt^2}{2} .$$

Combining last two equations we obtain

$$h - h_0 = v_B \cdot t + \frac{gt^2}{2} = v_B \cdot t + h - h_1 .$$

$$h_1 - h_0 = v_B \cdot t = \sqrt{2gh_0} \cdot t .$$

Solving for time t gives

$$t = \frac{h_1 - h_0}{\sqrt{2gh_0}} = \frac{15-5}{\sqrt{2 \cdot 9.8 \cdot 5}} = 1.01 \text{ s.}$$

The height of the top is

$$h = \frac{gt^2}{2} + h_1 = \frac{9.8 \cdot (1.01)^2}{2} + 15 = 20 \text{ m.}$$

Problem 1.20

A rocket is fired vertically with an upward acceleration of 25 *m/s². After* 20 *s, the engine shuts off and the rocket continues to move as a free particle until it reaches the ground. Calculate the highest point the rocket reaches, the total time the rocket is in the air, and the speed of the rocket just before it hits the ground.*

Solution

The first period of the rocket motion is upward accelerated motion with acceleration a =20 m/s^2 directed upwards. Since the rocket starts from rest, its initial velocity is zero, and the distance covered for 20 seconds is

$$h_1 = v_0 t_1 + \frac{at_1^2}{2} = \frac{at_1^2}{2} = \frac{25 \cdot 20^2}{2} = 5000 \text{ m.}$$

The velocity of the rocket at the end of this time interval is

$$v = v_0 + at = at = 25 \cdot 20 = 500 \text{ m/s.}$$

The second period of the rocket motion is upward decelerated motion with acceleration due to gravity $\vec{g}$ directed downwards. It takes until the velocity becomes zero ($v_2 = 0$) at the top point of the trajectory. The time interval from

$$0 = v - gt_2,$$

is equal to

$$t_2 = \frac{v}{g} = \frac{500}{9.8} = 51 \text{ s}.$$

The distance covered for this time is

$$h_2 = \frac{v_2^2 - v_1^2}{2a} = \frac{-v_1^2}{2 \cdot (-g)} = \frac{500^2}{2 \cdot 9.8} = 12755 \text{ m}.$$

The third period of the rocket motion is its free fall, i. e., the downward motion from rest with acceleration due to gravity $\vec{g}$ directed downwards. The height of the free fall is

$$H = h_1 + h_2 = 5000 + 12755 = 17755 \text{ m}.$$

The time of free fall is

$$t_3 = \sqrt{\frac{2H}{g}} = \sqrt{\frac{2 \cdot 17755}{9.8}} = 60.2 \text{ s}.$$

The total time the rocket is in the air

$$t = t_1 + t_2 + t_3 = 20 + 51 + 60.2 = 131.2 \text{ s}.$$

The velocity of the rocket just before it hits the ground is

$$v = gt = 9.8 \cdot 60.2 = 599 \text{ m/s}.$$

Problem 1.21

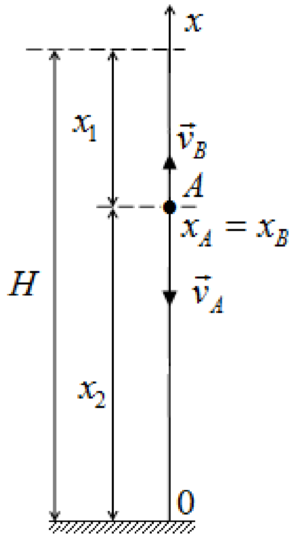
Ball A is dropped from the top of a building at the same instant that the ball B is thrown vertically upward from the ground. When the balls collide, they are moving in opposite directions, and the speed of A is twice the speed of B. a) At what fraction of the height of the building does the collision occur? b) Solve this problem if the collision occurs when the balls are moving in the same direction and the speed of A is 4 times that of B.

Solution

a) Take $x = 0$ at ground, upward direction is positive.

The distance covered by the first ball during its free falling to the point A for the time t is

$$x_1 = \frac{gt^2}{2}.$$



The coordinate of the collision point and the velocity of the first ball at this point are

$$x_A = H - x_1 = H - \frac{gt^2}{2},$$

$$v_A = -gt.$$

The coordinate of the collision point (x_B) determined for the second ball is separated from the origin by distance x_2,

$$x_B = x_2 = v_0 t - \frac{gt^2}{2},$$

and the velocity of the second ball at this point is

$$v_B = v_0 - gt.$$

Since the balls are moving in the opposite directions and the speed of the first ball is twice the speed of the second ball $v_A = -2v_B$, and

$$-gt = -2(v_0 - gt),$$

$$t = \frac{2v_0}{3g}.$$

From $x_A = x_B$,

$$H - \frac{gt^2}{2} = v_0 t - \frac{gt^2}{2},$$

$$H = v_0 t = \frac{2v_0^2}{3g}.$$

$$x_A = H - \frac{gt^2}{2} = \frac{2v_0^2}{3g} - \frac{g}{2}\left(\frac{2v_0}{3g}\right)^2 = \frac{4v_0^2}{9g}.$$

$$\frac{x_A}{H} = \frac{4v_0^2 \cdot 3g}{9g \cdot 2v_0^2} = \frac{2}{3}.$$

$$x_A = \frac{2H}{3}.$$

b) Now, for the second part of the problem, $v_A = 4v_B$. It gives

$$-gt = 4(v_0 - gt),$$

$$t = \frac{4v_0}{3g}.$$

From $x_A = x_B$, $\quad H - \frac{gt^2}{2} = v_0 t - \frac{gt^2}{2}$,

$$H = v_0 t = \frac{4v_0^2}{3g}.$$

$$x_A = H - \frac{gt^2}{2} = \frac{4v_0^2}{3g} - \frac{g}{2}\left(\frac{4v_0}{3g}\right)^2 = \frac{4v_0^2}{9g}.$$

$$\frac{x_A}{H} = \frac{4v_0^2 \cdot 3g}{9g \cdot 4v_0^2} = \frac{1}{3}.$$

$$x_A = \frac{H}{3}.$$

Problem 1.22

An object is thrown vertically and has an upward velocity of 5 *m/s when it reaches three fourths of its maximum height above its launch point. What is the initial (launch) speed of the object?*

Solution

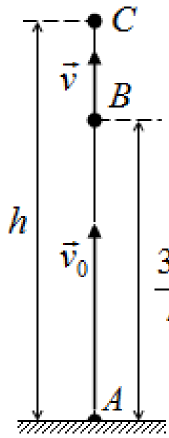
The velocity of the decelerated motion of the object depends on time as

$$v = v_0 - gt.$$

At the highest point C of the trajectory velocity of the object is zero

$0 = v_0 - gt$, and the time of the motion is $t = \frac{v_0}{g}$.

The maximum height of the object is



$$h = v_0 t - \frac{gt^2}{2} = \frac{v_0^2}{g}.$$

The height of the point B according to the given data is

$$h_B = \frac{3}{4}h = \frac{3}{4}\frac{v_0^2}{g}.$$

On the other hand, velocity of the object at the point B is

$$v = v_0 - gt_1,$$

and the time for reaching this point is

$$t_1 = \frac{v_0 - v}{g}.$$

Thus, the height of the point B is

$$h_B = v_0 t_1 - \frac{g t_1^2}{2} = \frac{v_0(v_0 - v)}{g} - \frac{g(v_0 - v)^2}{2g^2} = \frac{v_0^2 - v^2}{2g}.$$

Equating the expressions for the height of the point B, we obtain

$$\frac{3}{4}\frac{v_0^2}{g} = \frac{v_0^2 - v^2}{2g},$$

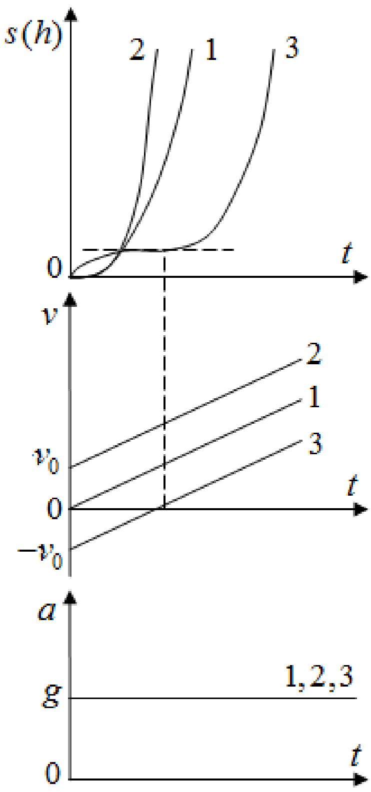
$$v_0 = 2 \cdot v = 2 \cdot 5 = 10 \text{ m/s}.$$

Problem 1.23

The object is dropped from a helicopter which is at the height h. Plot the acceleration-time, speed-time and distance-time graphs of the object motion if the helicopter a) is at rest; b) is ascending at uniform speed v_0; and c) is descending at the uniform velocity v_0.

Solution

a) The motion of the object dropped from the helicopter hovering at the height h is the free fall, i. e., the accelerated motion with constant acceleration g. Therefore, the acceleration-time graph is the straight line parallel to the horizontal axis. The velocity-time graph for uniformly accelerated motion is a straight line sloping upwards. Slope of velocity-time graph gives acceleration which is constant during the motion (acceleration due to gravity $g = 9.8$ m/s^2 in examined case). The distance depends upon the time in a quadratic way, therefore, the distance-time graph of a particle moving under the constant accelerated along straight line path is a parabola, more properly, rising branch of a parabola with initial point at $t = 0$ in the origin of the coordinate system.



The equations that describe this motion are

$$\begin{cases} a = g, \\ v = gt, \\ h = \dfrac{gt^2}{2}. \end{cases}$$

b) The object that was dropped from the helicopter descending with the constant directed downward velocity v_0 has the same initial downward velocity v_0. The equations describing this motion are

$$\begin{cases} a = g, \\ v = v_0 + gt, \\ h = v_0 t + \dfrac{gt^2}{2}. \end{cases}$$

The appropriate graphs are shown in the Figure and marked by number 2. Since the acceleration is $\vec{g}$, the slope of the velocity-time graph is the same as in the first case, but the line $v(t)$ is displaced by magnitude v_0 up from the origin. Parabolic dependence $s(t)$ becomes steeper with respect to the vertical axis in comparison to the free falling object.

c) The motion of the object dropped from the helicopter ascending with the constant velocity v_0 consists of two stages. Firstly, the object with directed upward velocity v_0 takes part in decelerated motion until its velocity becomes zero. After travelled the distance h_0, the object starts free falling from the height $h+h_0$. The equations for these stages of the object's motion are

$$\begin{cases} a = g, \\ v = -v_0 + gt, \\ h_0 = -v_0 t + \dfrac{gt^2}{2}. \end{cases} \qquad \begin{cases} a = g, \\ v = gt, \\ h + h_0 = \dfrac{gt^2}{2}. \end{cases}$$

The acceleration-time dependence (labeled by number 3) is the same as in two previous cases. The dependence $v(t)$ is displaced down from the origin by magnitude v_0 (to the point $-v_0$). The linear dependence cuts the horizontal axis at the instant of time when the object is at its highest point and its velocity is zero. The distance-time dependence consists of two parts with different curvatures: convex branch of parabola for decelerated (upward) motion and concave branch of parabola for accelerated (downward) motion. The flex point where the inflectional tangent to the curve is parallel to the horizontal axis corresponds to zero velocity at the top point of trajectory.

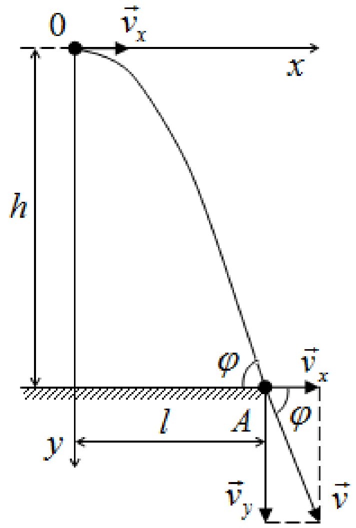
Problem 1.24

A stone is released from a height of 25 *m above the ground at the horizontal velocity* v_x =10 *m/s. How long does the stone take to strike the ground? With what velocity does the stone strike the ground? What is its horizontal displacement? What is the angle between the trajectory of the stone and the horizontal? Neglect the air resistance.*

Solution

A body which is in flight through the atmosphere but is not being propelled by any fuel is called projectile. The motion of projectile is a two-dimensional motion. So, it can be discussed in two parts: (i) horizontal motion, (ii) vertical motion. These two motions take place independent on each other. This called the principle of physical independence of motions.

The case described in this problem is the horizontal projectile, i. e., a body which is thrown horizontally from point O with horizontal velocity $\vec{v}_x$. The point O is at certain height *h* above the ground. Through the point O, take two axes –



x-axis and y-axis. Let x and y be the horizontal and vertical distances respectively covered by the projectile in time t.

The horizontal motion of the projectile is uniform motion. This is because the only force acting on the projectile is force of gravity. This force acts in the vertically downward direction, so the vertical motion is accelerated motion. The initial velocity in the vertically downward direction is zero, i. e., $v_{y0} = 0$. Since y-axis is taken downwards, therefore, the downward direction will be regarded as positive direction. So, the acceleration in vertical direction is $+g$.

The equations describing the motion are $\begin{cases} x = v_x t, \\ v_x = const, \\ y = \dfrac{gt^2}{2}, \\ v_y = gt. \end{cases}$

A body thrown horizontally from certain height above the ground follows a parabolic trajectory till it hits the ground in the point A.

For the point A the equations are

$$\begin{cases} l = v_x t, \\ h = \dfrac{gt^2}{2}, \\ v_y = gt. \end{cases}$$

From the second equation the time t of the motion along OA path is

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \cdot 25}{9.8}} = 2.26 \text{ s.}$$

Note that in the vertical motion, the displacement, velocity and acceleration are all vectors and thus have no effect in the perpendicular horizontal direction. Similarly the horizontal displacement and velocity have no effect in the perpendicular vertical direction. The only scalar quantity used to link the two independent motions is time. For as long as the body is falling, it is also moving horizontally and vice versa.

Therefore, we have the possibility of finding the distance l and the vertical component of velocity v_y at the point A:

$$l = 10 \cdot 2.26 = 22.6 \text{ m},$$

$$v_y = 9.8 \cdot 2.26 = 22.15 \text{ m/s}.$$

The velocity and its magnitude are

$$\vec{v} = \vec{v}_x + \vec{v}_y,$$

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{100 + 490.5} = 24.3 \text{ m/s}.$$

The angle φ, which the resultant velocity $\vec{v}$ makes with the horizontal, may be found from the relationship

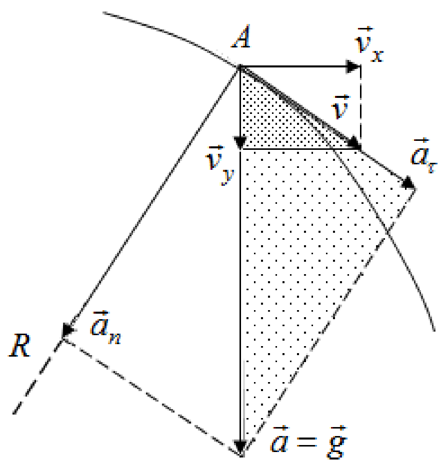
$$\tan\varphi = \frac{v_y}{v_x} = \frac{22.15}{10} = 2.215,$$

$$\varphi = 68.7°.$$

Problem 1.25

For the object thrown at horizontal velocity v_x = 10 m/s find the normal, tangential and total acceleration after time t = 2 s after the beginning of its motion and the radius of the curvilinear trajectory at this point.

Solution



Let us calculate the resultant velocity of the projectile at any point on the trajectory. The projectile reaches point A in time t. Let v_x and v_y be the horizontal and vertical components of $\vec{v}$. Since the horizontal motion of the projectile is uniform motion, v_x is not changed. The vertical motion is an accelerated motion. The y-component of velocity depends on time as

$$v_y = v_{y0} + gt\,.$$

Since $v_{y0} = 0$, the vertical component of the velocity is

$$v_y = gt = 9.8 \cdot 2 = 19.6 \text{ m/s}.$$

At any point of trajectory

$$\vec{v} = \vec{v}_x + \vec{v}_y\,.$$

The magnitude of resultant velocity $\vec{v}$ at the point A is given by

$$v = \sqrt{{v_x}^2 + {v_y}^2} = \sqrt{10^2 + 19.6^2} = 22 \text{ m/s}.$$

The resultant vector $\vec{v}$ and its components $\vec{v}_x$ and $\vec{v}_y$ form the triangle of velocities.

The horizontal motion is uniform motion. The vertical motion is accelerated motion with acceleration due to gravity $\vec{g}$ directed downwards. This acceleration may be resolved into two components mutually perpendicular to one another: tangential acceleration $\vec{a}_\tau$ directed along the resultant velocity, and normal acceleration $\vec{a}_n$ directed at right angle to the resultant velocity.

The resultant acceleration $\vec{g}$ and its components $\vec{a}_\tau$ and $\vec{a}_n$ form the triangle of accelerations, which is similar to the triangle of velocities (see figure to the problem). Therefore, their sides are proportional to each other

$$\frac{a}{v} = \frac{a_\tau}{v_y} = \frac{a_n}{v_x}\,.$$

From this proportion we can find

$$a_\tau = a\frac{v_y}{v} = g\frac{v_y}{v} = 9.8 \cdot \frac{19.6}{22} = 8.73 \text{ m/s}^2,$$

$$a_n = a\frac{v_x}{v} = g\frac{v_x}{v} = 9.8 \cdot \frac{10}{22} = 4.45 \text{ m/s}^2.$$

As the normal acceleration is $a_n = \dfrac{v^2}{R}$, the radius of the curvilinear trajectory is

$$R = \frac{v^2}{a_n} = \frac{22^2}{4.45} = 108.8 \text{ m}.$$

Problem 1.26

A ball is projected horizontally from a height at a speed of 30 *m/s. Find the time after which the vertical component of velocity becomes equal to horizontal component of velocity? (g* = 10 *m/s*^{2}*)*

Solution

The ball starts in horizontal direction, therefore, $\vec{v} = \vec{v}_x$. Due to the absence of horizontal forces (we neglect the air resistance) the horizontal component of velocity v_x remains constant. The vertical component of velocity is changing as $v_y = g \cdot t$ due to gravity.

At certain instant of time, the vertical component of velocity equals horizontal component of velocity

$$v_x = v_y,$$

$$v_x = g \cdot t,$$

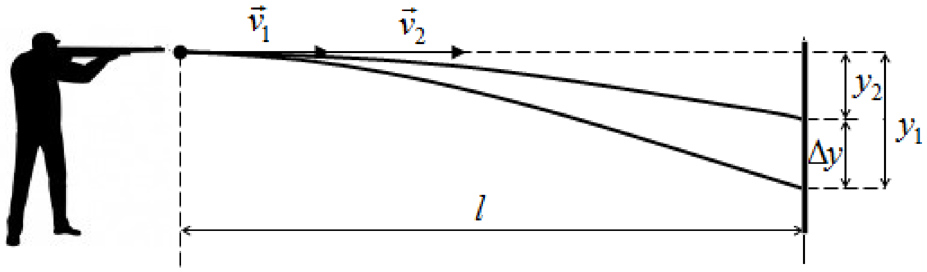
$$t = \frac{v_x}{g} = \frac{30}{10} = 3\,\text{s}.$$

Problem 1.27

Find the distance between the places of bullet hitting the target if the velocities of the bullets were v_1=320 *m/s and* v_2 = 350 *m/s and they were fired horizontally. The distance between the rifle and the target is l* = 60 *m*.

Solution

A bullet during its motion between the rifle and a target takes tart in two motions: horizontal uniform motion with $v_x = v$ and vertical accelerated motion under the action of gravity with $v_y = gt$. As a result, the bullet deflects from horizontal direction, and the magnitude of this deflection depends on the time of the bullet motion. This time may be determined as $t = \frac{l}{v_x}$ and is equal to $t_1 = \frac{l}{v_{x1}} = \frac{50}{320} = 0.156$ s for the first bullet, and $t_2 = \frac{l}{v_{x2}} = \frac{50}{350} = 0.143$ s for the second bullet.

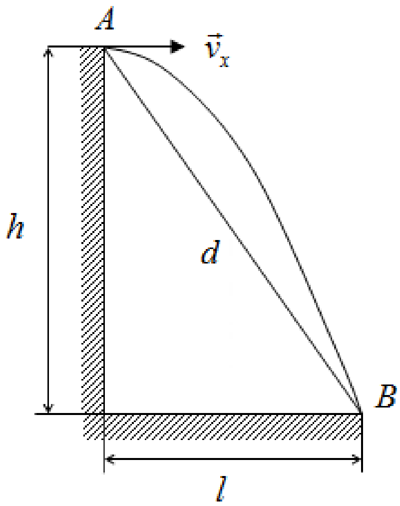


The deflections from the vertical direction are $y_1 = \frac{gt_1^2}{2}$ and $y_2 = \frac{gt_2^2}{2}$ respectively. Therefore, the distance between the holes is

$$\Delta y = y_1 - y_2 = \frac{g}{2}\left(t_1^2 - t_2^2\right) = \frac{9.8}{2}\left(0.156^2 - 0.143^2\right) = 1.89 \cdot 10^{-2} \text{ m}.$$

Problem 1.28

A tourist kicks a little stone horizontally off a cliff $h = 40$ m high into a river. If the tourist hears the sound of the splash $t_0 = 3$ s later, what was the initial speed given to the rock? Assume the speed of sound in air to be $v_s = 343$ m/s.



Solution

The stone's motion is described by the following equations

$$\begin{cases} l = v_x \cdot t, \\ h = \dfrac{gt^2}{2}. \end{cases}$$

The time of stone falling down is $t = \sqrt{2h/g}$.

$$l = v_x \cdot t = v_x \cdot \sqrt{2h/g}\ .$$

In contrast to the curvilinear path of the rock, the sound is propagating along the straight line BA = d, where

$$d = \sqrt{l^2 + h^2} = \sqrt{\frac{v_x^2 \cdot 2h}{g} + h^2}\ .$$

On the other hand, this is the distance which is covered by sound travelling with the velocity v_x for the time t_s

$$d = v_s \cdot t_s \ .$$

Combining two previous equations, we obtain

$$\sqrt{\frac{v_x^2 \cdot 2h}{g} + h^2} = v_s \cdot t_s ,$$

$$t_s = \frac{1}{v_s} \cdot \sqrt{\frac{v_x^2 \cdot 2h}{g} + h^2} \ .$$

Now, the total time $t_0 = t + t_s$ is

$$t_0 = \sqrt{\frac{2h}{g}} + \frac{1}{v_s} \cdot \sqrt{\frac{v_x^2 \cdot 2h}{g} + h^2} ,$$

$$v_s \cdot \left(t_0 - \sqrt{\frac{2h}{g}} \right) = \sqrt{\frac{v_x^2 \cdot 2h}{g} + h^2} ,$$

$$v_x^2 = \frac{g}{2h} \left(v_s^2 \cdot \left(t_0 - \sqrt{\frac{2h}{g}} \right)^2 - h^2 \right)$$

$$v_x = \sqrt{\frac{g}{2h} \cdot \left(v_s^2 \cdot \left(t - \sqrt{\frac{2h}{g}} \right)^2 - h^2 \right)} = \sqrt{\frac{9.8}{2 \cdot 40} \cdot \left(343^2 \cdot (3 - 2.86)^2 - 40^2 \right)} = 9.3 \text{ m/s.}$$

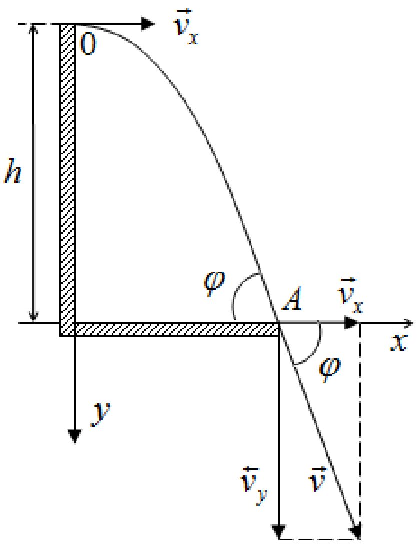
Problem 1.29

A marble is thrown horizontally with a speed of 10 *m/s from the top of a building. When it strikes the ground, the marble has a velocity that makes an angle of* 64° *with the horizontal. From what height above the ground was the marble thrown?*

Solution

Velocity triangle at the point A where the marble strikes the ground is formed by three vectors: the velocity $\vec{v}$, its vertical $\vec{v}_y$ and horizontal

$\vec{v}_x$ components. Assume that the angle that the trajectory makes with the ground is equal to the angle φ in the triangle formed by velocities.



$$\tan\varphi = \frac{v_y}{v_x},$$

$$v_y = v_x \cdot \tan\varphi.$$

On the other hand, during the downward motion the vertical component of velocity is changed as $v_y = gt$. The time of falling down is

$$t = \frac{v_y}{g}.$$

The distance traveled for this time and equaled to the height of the building is

$$h = \frac{gt^2}{2} = \frac{g}{2}\cdot\left(\frac{v_y}{g}\right)^2 = \frac{v_y^2}{2g} = \frac{(v_x \cdot \tan\varphi)^2}{2g} = \frac{(10\cdot\tan 64°)^2}{2\cdot 9.8} = 21.4 \text{ m}.$$

Problem 1.30

A stone is thrown at the initial velocity v_0 =10 *m/s at the angle* $\alpha = 30°$ *with the horizontal. Find the time of flight, the horizontal range of the projectile, and its maximum height.*

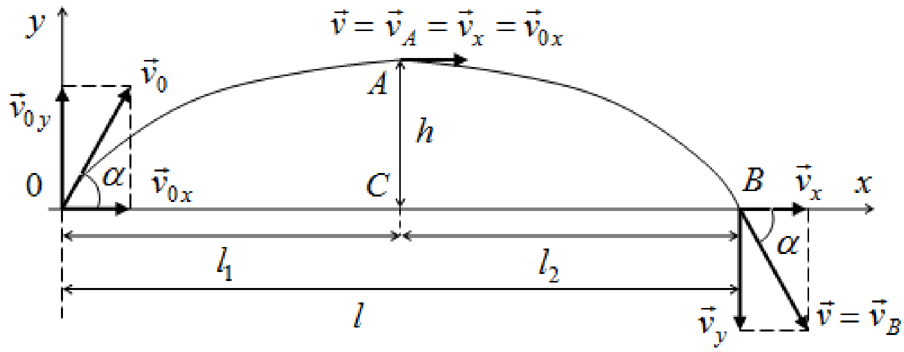
Solution

Consider a projectile thrown with velocity $\vec{v}_0$ at an angle α with horizontal (so called an *oblique projectile)*. Its velocity $\vec{v}_0$ can be resolved into two rectangular components along x-axis and y-axis, respectively,

$$\begin{cases} v_{0x} = v_0\cos\alpha, \\ v_{0y} = v_0\sin\alpha. \end{cases}$$

The motion of the projectile is a two-dimensional motion. It can be supposed to be made up of two motions – horizontal motion (along x-axis) and vertical motion (along y-axis). The horizontal motion of the projectile is uniform motion.

The only force acting on the projectile is the force of gravity; hence the vertical motion is accelerated motion. The projectile increases its height up to a maximum where its vertical velocity v_y becomes zero. After this, the projectile reverses its vertical direction and returns to earth striking the ground with a speed $\vec{v}_B$.



Let us consider the decelerated motion from the start point to the maximum height along the path OA during time t_1. It is described by the following system of equations

$$\begin{cases} x = v_{0x} t_1, \\ v_x = v_{0x}, \\ y = v_{0y} t - \dfrac{g t_1^2}{2}, \\ v_y = v_{0y} - g t_1. \end{cases}$$

At the point A, these equations take the following form

$$\begin{cases} l_1 = v_{0x} t_1, \\ h = v_{0y} t - \dfrac{g t_1^2}{2}, \\ 0 = v_{0y} - g t_1. \end{cases}$$

The time of motion from point O to the point A is

$$t_1 = \frac{v_{0y}}{g} = \frac{v_0 \sin\alpha}{g}.$$

The maximum height h and the distance l_1 are, respectively,

$$h = \frac{gt_1^2}{2} = \frac{v_0^2 \sin^2 \alpha}{2g},$$

$$l_1 = \frac{v_{0x} v_y}{g} = \frac{v_0^2 \cdot \sin\alpha \cdot \cos\alpha}{g} = \frac{v_0^2 \cdot \sin 2\alpha}{2g}.$$

Time of flight is the total time taken by the projectile to return to the same level from where it was thrown. Time of flight is equal to twice the time taken by the projectile to reach the maximum height. This is because the time of ascent is equal to the time of descent. This fact is also clear from symmetry of the parabolic curve along which the projectile is followed. So, time of flight

$$t = 2t_1 = \frac{2v_0 \sin\alpha}{g} = \frac{2 \cdot 10 \cdot 0.5}{9.8} = 1.02 \text{ s}.$$

The horizontal range is the total horizontal distance from the point of projection to the point where the projectile comes back to the plane of projection. This distance is covered for time $t = t_1 + t_2 = 2t_1$. Therefore, the horizontal range of the projectile is

$$l = 2l_1 = \frac{v_0^2 \cdot \sin 2\alpha}{g} = \frac{10^2 \sin 60°}{9,8} = 8,84 \text{ m}.$$

The height of the projectile flight is

$$h = \frac{v_0^2 \sin^2 \alpha}{2g} = \frac{10^2 \sin^2 30°}{2 \cdot 9,8} = 1,28 \text{ m}.$$

Problem 1.31

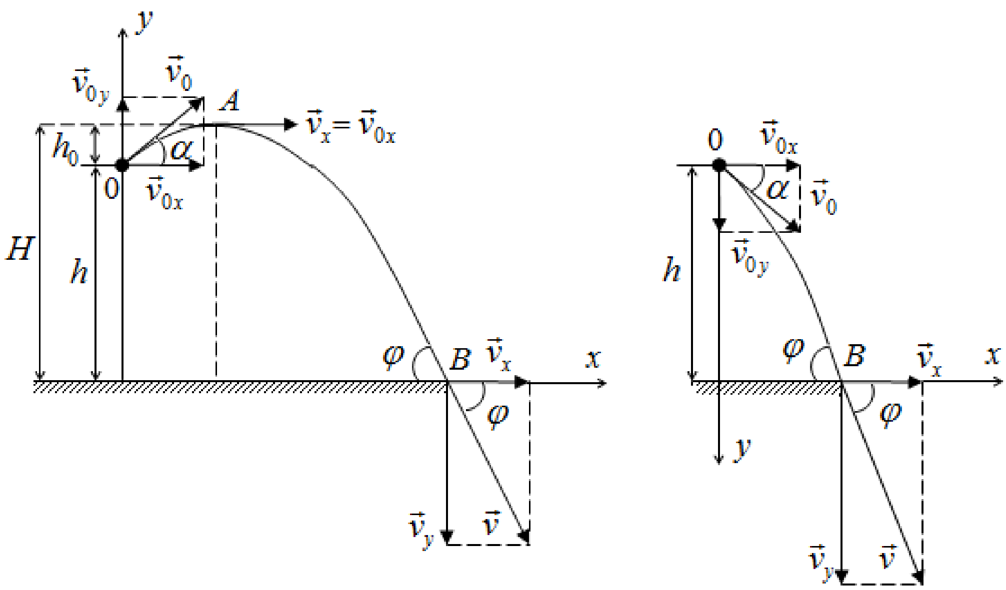
Two planes are about to drop an empty fuel tank. At the moment of release each plane has the same speed of 135 *m/s, and each tank is at the same height of around* 2 *km above the ground. Although the speeds are the same, the velocities are different at the instant of release, because the first plane is flying at an angle of* 15° *above the horizontal and the second is flying at an angle of* 15° *below the horizontal. Find the magnitude and the direction of the velocity with which the* *fuel tank hits the ground if it is from the first and from the second plane. Give the directional angles with respect to the horizontal.*

Solution

The velocities of the fuel tanks at the instant of release are the same as the velocities of the planes from which they are released; therefore, their paths are different.

The components of the velocity of the *first* plane (and the tank) are

$$\begin{cases} v_{0x} = v_0 \cdot \cos\alpha = 135 \cdot \cos 15° = 130.4 \text{ m/s}, \\ v_{0y} = v_0 \cdot \sin\alpha = 135 \cdot \sin 15° = 34.9 \text{ m/s}. \end{cases}$$



The vertical component v_{0y} is directed upwards. The upward motion of the tank is decelerated with acceleration $\vec{g}$ directed downwards.

$$v_y = v_{0y} - gt.$$

This motion takes place until the vertical component of velocity becomes zero.

$$0 = v_{0y} - gt.$$

The time for the motion is $t = \frac{v_{0y}}{g} = \frac{34.9}{9.8} = 3.56$ s.

The tank during this time ascended to the height

$$h_0 = v_{0y}t - \frac{gt^2}{2} = 34.9 \cdot 3.56 - \frac{9.8 \cdot 3.56^2}{2} \approx 62 \text{ m.}$$

The motion from the highest point (the point A) is accelerated motion from the height

$$H = h + h_0 = 2000 + 62 = 2062 \text{ m.}$$

The time of this motion is $t = \sqrt{\frac{2H}{g}} = \sqrt{\frac{2 \cdot 2062}{9.8}} = 20.5 \text{ s.}$

The vertical component of the velocity before hitting the ground is

$$v_y = gt = 9.8 \cdot 20.5 = 201 \text{ m/s.}$$

Taking into account that the horizontal component of velocity is not changed due to the absence of air resistance ($v_x = v_{0x} = 130.4$ m/s), the velocity of the tank at its landing is

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{130.4^2 + 201^2} = 239.6 \text{ m/s.}$$

The angle that the path makes with horizontal we can find from

$$\tan\varphi = \frac{v_y}{v_x} = \frac{201}{130.4} = 1.54,$$

$$\varphi = \arctan 1.54 = 57°.$$

The motion of the *second* tank is accelerated motion with the initial vertical velocity $v_{0y} = v_0 \cdot \sin\alpha = 135 \cdot \sin 15° = 34.9$ m/s directed downwards. The horizontal component of velocity is constant due to the absence of the air resistance

$$v_x = v_{0x} = v_0 \cdot \cos\alpha = 135 \cdot \cos 15° = 130.4 \text{ m/s}.$$

The tank starts at the height $h = 2000$ m and its flight continues during

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \cdot 2000}{9.8}} = 20.2 \text{ s.}$$

The vertical component of tank velocity is

$$v_y = v_{0y} + gt = 34.9 + 9.8 \cdot 20.2 = 232.9 \text{ m/s}.$$

The velocity of the tank from the second plane near the ground is

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{130.4^2 + 232.9^2} = 266.9 \text{ m/s}.$$

The angle that the path makes with horizontal we can find from

$$\tan\varphi = \frac{v_y}{v_x} = \frac{266.9}{130.4} = 2.05,$$

$$\varphi = \arctan 2.05 = 64°.$$

Problem 1.32

By trial and error, a frog learns that it can leap a maximum horizontal distance of 1.4 *m. If, in the course of an hour, the frog spends* 32% *of the time resting and* 68% *of the time performing identical jumps of that maximum length, in a straight line, what is the distance traveled by the frog?*

Solution

The range of each projectile (including the frog) is

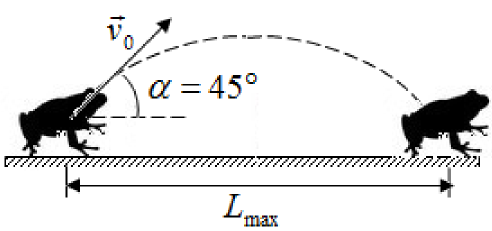
$$L = \frac{v_0^2 \cdot \sin 2\alpha}{g}.$$

The maximum range is at $\alpha = 45°$ and is equal to

$$L_{\max} = \frac{v_0^2}{g}.$$

The initial velocity of the jumping frog has to be

$$v_0 = \sqrt{\frac{L_{\max}}{g}} = \sqrt{\frac{1.4}{9.8}} = 3.7 \text{ m/s}.$$



The time of one jump is

$$t = \frac{2v_0 \sin\alpha}{g} = \frac{2 \cdot 3.7 \cdot \sin 45°}{9.8} = 0.53 \text{ s}.$$

Taking into account that the frog is jumping during 0.68 min each hour, i. e., $0.68 \cdot 3600 = 2448$ seconds, we can determine the number of its jumps as $N = 2448 / 0.53 = 4619$.

The total distance is $s = N \cdot L_{\max} = 4619 \cdot 1.4 = 6467$ m.

Problem 1.33

A trained dolphin leaps from the water with an initial speed of 12 m/s. It jumps directly towards a ball held by the trainer a horizontal distance of 5.5 *m away and a vertical distance of* 4.1 *m above the water. In the absence of gravity the dolphin would move in a straight line to the ball and catch it, but because of gravity the dolphin follows a parabolic path well below the ball's initial position. If the trainer releases the ball the instant the dolphin leaves the water, show that the dolphin and the falling ball meet.*

Solution

The angle that the dolphin leaves the water is

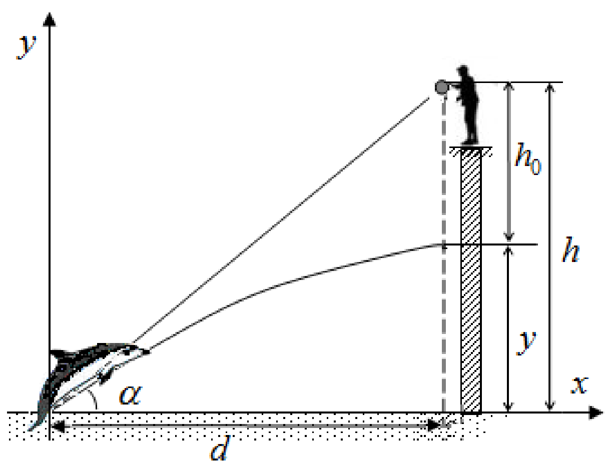
$$\tan\alpha = \frac{h}{d} = \frac{4.1}{5.5} = 0.745,$$

$$\alpha = \arctan 0.745 = 36.7°.$$

The horizontal and vertical components of the initial velocity of the dolphin are

$$v_{0x} = v_0 \cdot \cos\alpha = 12 \cdot \cos 36.7° = 9.62 \text{ m/s},$$

$$v_{0y} = v_0 \cdot \sin\alpha = 12 \cdot \sin 36.7° = 7.17 \text{ m/s}.$$



Since the air resistance is neglected, horizontal component $v_x = v_{0x}$.

The distance covered in horizontal direction is

$$x = v_x \cdot t,$$

and the time of the dolphin motion is

$$t = \frac{x}{v_x} = \frac{5.5}{9.62} = 0.572 \text{ s}.$$

The vertical displacement of the dolphin is

$$y = v_{0y} \cdot t - \frac{gt^2}{2} = 7.17 \cdot 0.572 - \frac{9.8 \cdot 0.572^2}{2} = 2.5 \text{ m}.$$

The ball's location at this instant of time is

$$y = h - h_0 = h - \frac{gt^2}{2} = 4.1 - \frac{9.8 \cdot 0.572^2}{2} = 2.5 \text{ m}.$$

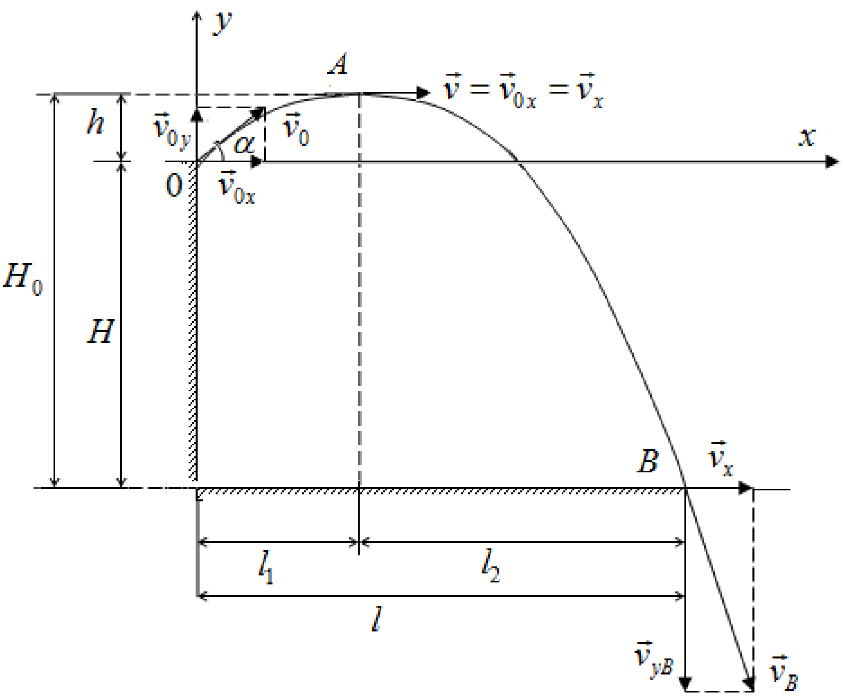
The calculation shows that after 0.572 s the dolphin and the ball are at the same point at the height 2.5 m above the water surface.

Problem 1.34

A projectile is thrown from the top of a building 30 *m high, at an angle of* 40° *with the horizontal speed of* 20 *m/s. Find a) time of the flight; b) horizontal distance covered at the end of journey; c) the maximum height of the projectile above the ground; d) Find the magnitude of the final velocity.*

Solution

Let's take the origin of coordinate system at the projectile starting point O.



The components of the initial velocity $\vec{v}_0$ are

$$\begin{cases} v_{0x} = v_0 \cdot \cos\alpha = 20 \cdot \cos 40° = 15.32 \quad \text{m/s}, \\ v_{0y} = v_0 \cdot \sin\alpha = 20 \cdot \sin 40° = 12.86 \quad \text{m/s}. \end{cases}$$

Firstly we examine the motion along OA path, which can be resolved by to simple motions: horizontal uniform motion at the velocity $v_x = v_{0x}$, and decelerated vertical motion, where the vertical component of velocity and height are given by the equations for a motion with uniform acceleration g.

$$\begin{cases} x = v_x \cdot t, \\ y = v_{0y} \cdot t - \dfrac{g \cdot t^2}{2}, \\ v_y = v_{0y} - g \cdot t. \end{cases}$$

For the point A these equations give

$$\begin{cases} l_1 = v_x \cdot t_1, \\ h = v_{0y} \cdot t_1 - \dfrac{g \cdot t_1^2}{2}, \\ 0 = v_{0y} - g \cdot t_1. \end{cases}$$

Now, $t_1 = \dfrac{v_{0y}}{g} = \dfrac{12.86}{9.8} = 1.31$ s,

$$h = v_{0y} \cdot \frac{v_{0y}}{g} - \frac{g \cdot v_{0y}^2}{2 \cdot g^2} = \frac{v_{0y}^2}{2 \cdot g} = \frac{12.86^2}{2 \cdot 9.8} = 8.44 \text{ m},$$

$$l_1 = v_x \cdot t_1 = \frac{v_{0x} \cdot v_{0y}}{g} = \frac{v_0^2 \cdot \sin 2\alpha}{2 \cdot g} = \frac{12.86^2 \cdot \sin 80°}{2 \cdot 9.8} = 8.31 \text{ m}.$$

For the accelerated motion along AB path

$$\begin{cases} x = v_x \cdot t_2, \\ y = \dfrac{g \cdot t_2^2}{2}, \\ v_y = g \cdot t_2. \end{cases}$$

The height of the point A is $H_0 = h + H = 8.44 + 30 = 38.44$ m.

For the point B these equation are transformed in

$$\begin{cases} l_2 = v_x \cdot t_2, \\ H_0 = \dfrac{g \cdot t_2^{\,2}}{2}, \\ v_{yB} = g \cdot t_2. \end{cases}$$

The time of descending is $t_2 = \sqrt{\dfrac{2H_0}{g}} = \sqrt{\dfrac{2 \cdot 38.44}{9.8}} = 2.8$ s,

$$l_2 = v_x \cdot t_2 = 15.32 \cdot 2.8 = 42.9 \text{ m},$$

$$v_{yB} = g \cdot t_2 = 9.8 \cdot 2.8 = 27.44 \text{ m/s}.$$

Since the final velocity at the point B is $\vec{v}_B = \vec{v}_x + \vec{v}_{yB}$, its magnitude is equal to

$$v_B = \sqrt{v_x^2 + v_{yB}^2} = \sqrt{15.32^2 + 27.44^2} = 31.43 \text{ m/s}.$$

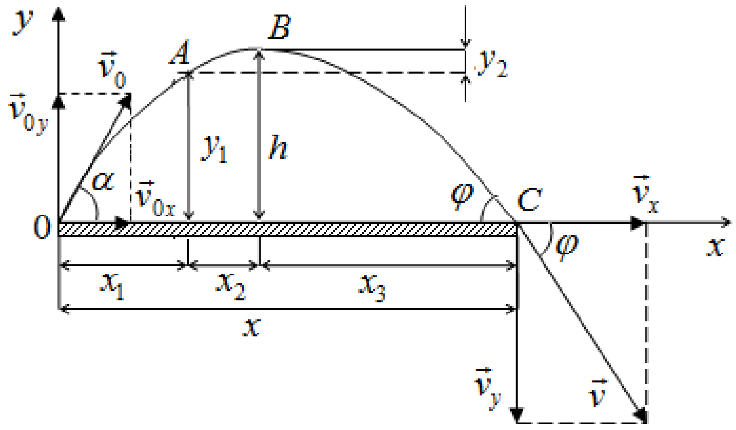
Problem 1.35

A catapult launches a rocket at an angle of 50° *above the horizontal with an initial speed of* 110 *m/s. The rocket engine immediately starts a burn, and for* 2.5 *s the rocket moves along its initial line of motion with an acceleration of* 35 *m/s*^{2}*. Then its engine fails, and the rocket proceeds to move in free-fall. Find a) the maximum altitude reached by the rocket; b) the total time of flight; c) the horizontal range; d) the velocity of the rocket at hitting the ground; e) the angle that the path makes with horizontal axis at the point of the rocket landing.*

Solution

The projectile moves in a curved path (parabola) and it leaves the origin (the point O) with velocity $\vec{v}$ which horizontal and vertical components are

$$\begin{cases} v_{0x} = v_0 \cdot \cos\alpha = 110 \cdot \cos 50° = 70.7 \text{ m/s}, \\ v_{0y} = v_0 \cdot \sin\alpha = 110 \cdot \sin 50° = 84.26 \text{ m/s}. \end{cases}$$



The motion of the rocket consists of three parts: upwards motion with engine (OA); upward motion without engine (AB); and downward motion (BC).

a) During the first stage the rocket is moving at acceleration $\vec{a}$. The components of acceleration are

$$\begin{cases} a_x = a \cdot \cos\alpha = 35 \cdot \cos 50° = 22.5 \text{ m/s}^2, \\ a_y = a \cdot \sin\alpha = 35 \cdot \sin 50° = 26.8 \text{ m/s}^2. \end{cases}$$

At the instant when the engine fails the components of the rocket velocity are

$$\begin{cases} v_x = v_{0x} + a_x t = 70.7 + 22.5 \cdot 2.5 = 126.95 \text{ m/s}, \\ v_y = v_{0y} + a_y t = 84.26 + 26.8 \cdot 2.5 = 113.56 \text{ m/s}. \end{cases}$$

Coordinates of the rocket at this instant of time (point A) are

$$\begin{cases} x_1 = v_{0x} \cdot t + \dfrac{a_x \cdot t^2}{2} = 70.7 \cdot 2.5 + \dfrac{22.5 \cdot 2.5^2}{2} = 247.05 \text{ m}, \\ y_1 = v_{0y} \cdot t + \dfrac{a_y \cdot t^2}{2} = 84.26 \cdot 2.5 + \dfrac{26.8 \cdot 2.5^2}{2} = 294.4 \text{ m}. \end{cases}$$

b) The upward motion along parabolic path to the highest point B. This is decelerated motion with acceleration due to gravity – $\vec{g}$ (directed downwards). For this motion $a_x = a_y = a = 0$; the velocity in horizontal direction $v_x = 126.95$ m/s and it is not changed during motion; vertical component of velocity

$$v_{y1} = v_y - g t_1.$$

At the highest point of trajectory (point B) $v_{y1}=0$, therefore,

$$t_1=\frac{v_y}{g}=\frac{113.56}{9.8}=11.59 \text{ s.}$$

$$\begin{cases} x_2=v_x\cdot t=126.95\cdot 11.59=1471\,\text{m}, \\ y_2=v_y\cdot t_1-\dfrac{g\cdot t_1^2}{2}=113.56\cdot 11.59-\dfrac{9.8\cdot 11.59^2}{2}=658\,\text{m}. \end{cases}$$

c) Downward accelerated motion from the height h

$$h=y_1+y_2=294.4+658=952.4 \text{ m.}$$

$$v_{y2}=\sqrt{2gh}=\sqrt{2\cdot 9.8\cdot 952.4}=\sqrt{18666}=136.6 \text{ m/s,}$$

Since $v_x=126.95$ m/s, and $\vec{v}=\vec{v}_x+\vec{v}_{y2}$, the magnitude of velocity of the rocket at the point C where it hits the ground is

$$v=\sqrt{v_x^2+v_{y2}^2}=\sqrt{126.95^2+136.6^2}=186.5 \text{ m/s.}$$

$$v_{y2}=g\cdot t_2.$$

$$t_2=\frac{v_{y2}}{g}=\frac{136.6}{9.8}=13.94\,\text{s.}$$

$$x_3=v_x\cdot t_3=126.95\cdot 13.94=1769.5\,\text{m.}$$

The total time of the rocket motion is

$$t_0=t+t_1+t_2=2.5+11.59+13.94=28.03 \text{ s.}$$

The horizontal range of the projectile is equal to

$$x_0=x_1+x_2+x_3=247.05+1471+1769.5=3487.55 \text{ m.}$$

d) The angle that the path makes with horizontal axis at rocket landing is

$$\tan\varphi=\frac{v_{y2}}{v_x}=\frac{136.6}{126.95}=1.08,$$

$$\varphi=47^\circ.$$

Problem 1.36

Find the angular speed a) of the second hand on clock; b) of the minute hand on clock; c) of the hour hand on clock; d) of the Earth's rotation about its axis.

Solution

a) The angular speed of the second hand on clock is

$$\omega = \frac{2\pi}{60} = \frac{\pi}{30} = 0.1 \text{ rad/s.}$$

b) The angular speed of the minute hand on clock is

$$\omega = \frac{2\pi}{60 \cdot 60} = \frac{\pi}{1800} = 0.0017 \text{ rad/s.}$$

c) The angular speed of the hour hand on clock is

$$\omega = \frac{2\pi}{12 \cdot 3600} = \frac{\pi}{21600} = 1.45 \cdot 10^{-4} \text{ rad/s.}$$

d) The Earth is rotating about its axis. It takes 1 day to complete one rotation, during which the angular displacement is 2π radians. So the angular speed of the Earth is 2π per 1 day.

1 day = 24 hours =24·3600 s =86400 s

The angular speed of the earth's rotation about its axis is

$$\omega = \frac{2\pi}{86400} = \frac{\pi}{43200} = 7.27 \cdot 10^{-5} \text{ rad/s.}$$

Problem 1.37

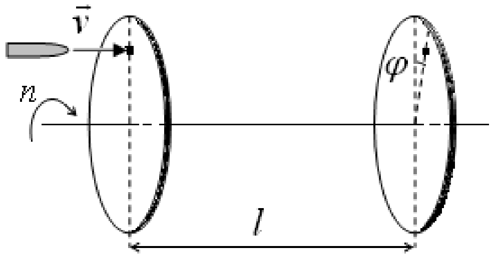
The axis with two disks located 0.5 *m apart from each other, rotates at a frequency* 1600 *rev/min. The bullet flying along an axis perforates both disks; thus the bullet hole in the second disk is misaligned with respect to the first disk by the angle* $\varphi = 12°$. *Find the bullet velocity.*

Solution

To find the bullet velocity we have to know the time taken to of cover the distance l.

If the frequency of disc rotation is n its angular speed is $\omega = 2\pi n$. The magnitude of angular displacement at this uniform rotational motion is

$$\varphi = \omega t = 2\pi n t,$$



and the time is

$$t = \frac{\varphi}{2\pi n}.$$

Substituting $\varphi = 12° = \frac{\pi}{15}$ rad, $n = 1600$ rev/min $= \frac{1600}{60}$ rev/s, we obtain

$$t = \frac{\cancel{\pi} \cdot 60}{15 \cdot 2\cancel{\pi} \cdot 1600} = 1.25 \cdot 10^{-3}\ \text{s}.$$

Then the velocity of the bullet is

$$v = \frac{l}{t} = \frac{0.5}{1.25 \cdot 10^{-3}} = 400\ \text{m/s}.$$

Problem 1.38

A wheel begins to rotate with uniform acceleration $\varepsilon =$ 3 *rad/s²*. *Find the angular velocity which the wheel reaches after* 3 *s of rotation. Determine the number of revolutions for this period of time.*

Solution

If the wheel takes part in the rotation with uniform acceleration its motion may be described by the system of equations

$$\begin{cases} \varphi = \omega_0 \cdot t + \dfrac{\varepsilon \cdot t^2}{2}, \\ \omega = \omega_0 + \varepsilon \cdot t. \end{cases}$$

Since the wheel starts from rest, its initial angular velocity is $\omega_0 = 0$, therefore,

$$\begin{cases} \varphi = \dfrac{\varepsilon \cdot t^2}{2}, \\ \omega = \varepsilon \cdot t. \end{cases}$$

Consequently, the angular velocity is

$$\omega = \varepsilon \cdot t = 3 \cdot 3 = 9 \text{ rad/s},$$

and the number of revolutions for this period of time is

$$N = \frac{\varphi}{2\pi} = \frac{\varepsilon t^2}{4\pi} = \frac{3 \cdot 3^2}{4\pi} = 2.15 \text{ rev.}$$

Problem 1.39

A fan blade spins at the frequency of 900 *rev/min. After switching off it was stopped after* $N = 75$ *revolutions. What is its angular acceleration? How long does the fan take to stop*?

Solution

The decelerated motion of the fan is described by

$$\begin{cases} 2\pi \cdot N = 2\pi \cdot n_0 \cdot t - \dfrac{\varepsilon \cdot t^2}{2}, \\ 2\pi \cdot n = 2\pi \cdot n_0 - \varepsilon \cdot t. \end{cases}$$

The final velocity is $n = 0$, then the second equation of system is

$$0 = 2\pi \cdot n_0 - \varepsilon \cdot t,$$

and the time of the motion is $t = \dfrac{2\pi \cdot n_0}{\varepsilon}$. Substituting time in the first equation and taking into account that $n_0 = 900$ rev/min = 15 rev/s, we obtain

$$\varepsilon = \frac{\pi \cdot n_0^2}{N} = \frac{\pi \cdot 15^2}{75} = 9.42 \text{ rad/s}^2.$$

The time of motion is

$$t = \frac{2\pi \cdot n_0}{\varepsilon} = \frac{2\pi \cdot 15}{9.42} = 10 \text{ s}.$$

Problem 1.40

A wheel starting from rest is uniformly accelerated at 4 *rad/s*^{2} *for* 5 *seconds. It is allowed to rotate uniformly for next* 50 *seconds and is finally brought to rest in the next* 10 *seconds. Find the total angle rotated by the wheel.*

Solution

There are three periods of the wheel rotation. Accelerated motion during $t_1 = 5\,\text{s}$ is described as

$$\begin{cases} \omega = \varepsilon_1 \cdot t_1 = 4 \cdot 5 = 20 \quad \text{rad/s}, \\ \varphi_1 = \dfrac{\varepsilon_1 \cdot t_1^2}{2} = \dfrac{4 \cdot 5^2}{2} = 50 \quad \text{rad}. \end{cases}$$

The angle rotated during $t_2 = 50$ seconds of the uniform motion at the angular velocity $\omega = 20$ rad/s is

$$\varphi_2 = \omega \cdot t_2 = 20 \cdot 50 = 1000 \ \text{rad}.$$

During next 10 seconds of decelerated motion the magnitude of acceleration may be calculated using equation $\omega = \omega - \varepsilon_2 \cdot t_3$. Since $\omega = 0$,

$$0 = \omega - \varepsilon_2 \cdot t_3,$$

$$\varepsilon_2 = \frac{\omega}{t_3} = \frac{20}{10} = 2 \ \text{rad/s}^2.$$

The angle rotated for this time is

$$\varphi_3 = \omega \cdot t_3 - \frac{\varepsilon_2 \cdot t_3}{2} = 20 \cdot 10 - \frac{2 \cdot 10^2}{2} = 100 \ \text{rad}.$$

The total angle is

$$\varphi = \varphi_1 + \varphi_2 + \varphi_3 = 50 + 1000 + 100 = 1150 \ \text{rad}.$$

Problem 1.41

The point is rotating along the circular path of radius R = 20 *cm with the uniform tangential acceleration* $a_\tau = 5$ *cm/s*^{2}. *Find the period of time until the instant when the normal acceleration is twice greater than the tangential acceleration.*

Solution

The angular velocity of the point during the accelerated motion may be calculated from the relationship

$$\omega = \omega_0 + \varepsilon \cdot t.$$

As $\omega_0 = 0$, then

$$\omega = \varepsilon \cdot t.$$

The normal acceleration

$$a_n = \omega^2 \cdot R = \left(\varepsilon \cdot t\right)^2 \cdot R.$$

The tangential acceleration is

$$a_\tau = \varepsilon \cdot R.$$

Since the normal acceleration is twice greater than the tangential acceleration $a_n = 2a_\tau$ we may write

$$\left(\varepsilon t\right)^2 R = 2\varepsilon \cdot R.$$

Consequently,

$$t = \sqrt{\frac{2}{\varepsilon}} = \sqrt{\frac{2R}{a_\tau}} = \sqrt{\frac{2 \cdot 0.2}{0.05}} = 2.83 \text{ s.}$$

Problem 1.42

The point is moving along the circular path of radius $R = 2$ *cm. The dependence of the distance on time is* $s(t) = Ct^3$, *where C* = 0.1 *cm/s*^{3}. *Find the normal and tangential accelerations of the point in the instant of time when the linear velocity of the point is* $v = 0.3$ *m/s*.

Solution

Using the dependence of the distance on time $s(t) = Ct^3$, we can find the dependencies of velocity and tangential acceleration on time

$$v = \frac{ds}{dt} = 3Ct^2,$$

$$a_\tau = \frac{dv}{dt} = 6Ct.$$

Therefore,

$$t = \sqrt{\frac{v}{3C}} = \sqrt{\frac{0.3}{3 \cdot 0.1 \cdot 10^{-2}}} = 10 \text{ s}.$$

The tangential acceleration is

$$a_\tau = 6 \cdot Ct = 6 \cdot 0.1 \cdot 10^{-2} \cdot 10 = 0.06 \text{ m/s}^2.$$

The normal acceleration

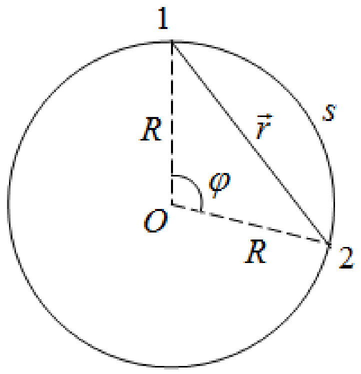
$$a_n = \frac{v^2}{R} = \frac{0.3^2}{2 \cdot 10^{-2}} = 4.5 \text{ m/s}^2.$$

Problem 1.43

The point is moving along the circular path of radius $R = 4$ *m. The initial velocity of the point is* $v_0 = 3$ *m/s, tangential acceleration* $a_\tau = 1$ *m/s*^{2}. *Find for the time instant* $t = 2$ *s: a) the distance covered for this time; b) the magnitude of displacement; c) the linear and angular velocities; d) the normal, total and angular accelerations.*

Solution

The dependence of the distance covered by the point on time is



$$s(t) = v_0 \cdot t + \frac{a_\tau \cdot t^2}{2} \text{ m.}$$

It allows finding the distance

$$s = 3 \cdot 2 + \frac{1 \cdot 2^2}{2} = 8 \text{ m.}$$

Taking into account that during one revolution the point covers the distance which is equal to the length of the circle

$$s_1 = 2\pi \cdot R = 8\pi \text{ (m),}$$

we can find the angular displacement from the proportion

$$\frac{2\pi}{\varphi} = \frac{8\pi}{8}, \ \varphi = 2 \text{(rad)} = 114.7°.$$

Therefore, the magnitude of displacement as the chord related to the angle φ may be calculated according to cosine theorem:

$$|\vec{r}| = \sqrt{2R^2 - 2R^2 \cdot \cos\varphi} = R\sqrt{2(1 - \cos\varphi)} = 4\sqrt{2(1 + 0.418)} = 6.73 \text{ m.}$$

The linear velocity of the point is

$$v = v_0 + a_\tau \cdot t = 3 + 1 \cdot 2 = 5 \text{ m/s.}$$

The angular velocity is

$$\omega = v \cdot R = 5 \cdot 4 = 20 \text{ rad/s.}$$

The normal acceleration is

$$a_n = \frac{v^2}{R} = \frac{5^2}{4} = 6,25 \text{ m/s}^2.$$

The total acceleration is

$$\vec{a} = \vec{a}_n + \vec{a}_\tau,$$

and its magnitude is

$$a=\sqrt{a_n^2+a_\tau^2}=\sqrt{6.25^2+1^2}=6.33\ \text{m/s}^2.$$

The angular acceleration is

$$\varepsilon=\frac{a_\tau}{R}=\frac{1}{4}=0.25\ \text{rad/s}^2.$$

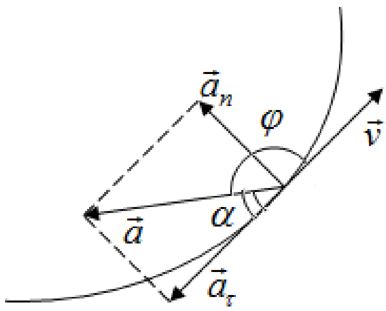
Problem 1.44

The car is moving at the velocity 36 *km/h along the curvilinear road of radius R = 200 m. It begins to decelerate with the acceleration* 0.3 *m/s*^{2}*. Find the normal acceleration of the car. Find the angle* φ *between the vector of total acceleration and the vector of the velocity. Find the angular velocity and acceleration of the car at the beginning of turning.*

Solution

If the velocity of the car is $v = 36$ km/h = 10 m/s, the normal acceleration is

$$a_n=\frac{v^2}{R}=\frac{10^2}{200}=0.5\ \text{m/s}^2.$$



The total acceleration of the car

$$a=\sqrt{a_n^2+a_\tau^2}=\sqrt{0.5^2+0.3^2}=0.58\ \text{m/s}^2.$$

The angular acceleration

$$\varepsilon=\frac{a_\tau}{R}=\frac{0.3}{200}=1.5\cdot10^{-3}\ \text{m/s}^2.$$

The angular velocity

$$\omega=\frac{v}{R}=\frac{10}{200}=0.05\ \text{rad/s}.$$

As the car takes part in decelerated motion, the vectors of velocity $\vec{v}$ and tangential acceleration $\vec{a}_\tau$ are directed in the opposite directions, and total acceleration and velocity make the obtuse angle φ. Therefore, after finding the supplementary angle α

$$\tan\alpha = \frac{a_n}{a_\tau} = \frac{0.5}{0.3} = 1.67,$$

$$\alpha = \arctan 1.67 = 59°,$$

we can determine the required angle

$$\varphi = 180° - \alpha = 180° - 59° = 121°.$$

Problem 1.45

The point is moving along the circle of radius R = 20 cm with constant tangential acceleration a_τ. Find normal a_n, tangential a_τ and total acceleration a of the point after t =20 s of the motion if it is known that in the end of the fifth revolution after the beginning of the motion the linear velocity of the point was v = 10 cm/s.

Solution

There are two instants of time that are considered in this problem: the first instant t_1 when the point has made 5 revolutions, and the second instant – t_2 after 20 s of motion.

The point takes part in accelerated motion with initial velocity equaled to zero, and then its motion is described by the equations

$$\begin{cases} 2\pi \cdot N = \dfrac{\varepsilon \cdot t_1^2}{2}, \\ 2\pi \cdot n = \varepsilon \cdot t_1, \end{cases}$$

or, taking into account that $2\pi \cdot n = \omega = \dfrac{v}{R}$,

$$\begin{cases} 2\pi \cdot N = \dfrac{\varepsilon \cdot t_1^2}{2}, \\ \dfrac{v}{R} = \varepsilon \cdot t_1. \end{cases}$$

Let us take the square of the second equation of the system, and divide the result termwise by the first equation

$$\frac{2\pi\cdot N\cdot R^2}{v^2}=\frac{\cancel{\varepsilon}\cdot\cancel{t_1}^2}{\varepsilon^{\cancel{2}}\cdot\cancel{t_1}^2\cdot 2}.$$

Then the angular acceleration is

$$\varepsilon=\frac{v^2}{4\pi\cdot N\cdot R^2},$$

and the tangential acceleration which is independent on time is

$$a_\tau=\varepsilon\cdot R=\frac{v^2}{4\pi\cdot N\cdot R}=\frac{10^{-2}}{4\pi\cdot 5\cdot 0.2}=8\cdot 10^{-4}\ \text{m/s}^2.$$

The normal acceleration which depends on velocity for the instant of time t_2 is

$$a_n=\omega_2^2\cdot R=\left(\varepsilon\cdot t_2\right)^2\cdot R=\frac{v^4\cdot t_2^2\cdot\cancel{R}}{16\pi^2\cdot N^2\cdot R^{\cancel{4}3}}=$$

$$=\frac{v^4\cdot t_2^2}{16\pi^2\cdot N^2\cdot R^3}=\frac{10^{-4}\cdot 400}{16\pi^2\cdot 25\cdot 8\cdot 10^{-3}}=3.2\cdot 10^{-2}\ \text{m/s}^2.$$

CONTROL PROBLEMS

1. A runner runs 2.5 km in 9 min and then takes 30 min to walk back to the starting point. *a*) What is the runner's average velocity for the first 9 min? *b*) What is the average velocity for the time spent walking? *c*) What is the average velocity for the whole trip? *d*) What is the average speed for the whole trip? [16.7 km/h; – 5 km/hr; 0; 7.7 km/h]

2. A race car's velocity increases from 4m/s to 36 m/s over a 4 s time interval. Then the car slows from 36 m/s to 15 m/s over 3 s. What is its average acceleration for each time interval? [8 m/s^2; – 7 m/s^2]

3. A bus is moving at 25 m/s when the driver steps on the brakes and brings the bus to a stop in 3 s. What is the average acceleration of the bus while braking? If the bus took twice as long to stop, how would the acceleration compare with what you found in previous case? [– 8.3 m/s^2; – 4.17 m/s^2]

4. A bus that is traveling at 30 km/h speeds up at a constant rate of 3.5 m/s^2. What velocity does it reach 6.8 s later? [120 km/h]

5. If a car accelerates from rest at a constant acceleration 5.5 m/s^2, how long will it take for the car to reach a velocity of 28 m/s? [5.1 s]

6. A car slows from 22 m/s to 3 m/s at a constant rate of 2.1 m/s^2. How many seconds are required before the car is traveling at 3 m/s? [9 s]

7. A man runs at a velocity of 4.5 m/s for 15 min. When going up an increasingly steep hill, he slows down at a constant rate of 0.05 m/s^2 for 90 s and comes to a stop. How far did he run? [$4.3 \cdot 10^3$ m]

8. A woman driving at a speed of 23 m/s sees a deer on the road ahead and applies the brakes when she is 210 m from the deer. If the deer does not move and the car stops right before it hits the deer, what is the acceleration provided by the car's brakes? [–1.3 m/s^2]

9. An in-line skater first accelerates from rest to 5 m/s in 4.5 s, and then continues at this constant speed for another 4.5 s. What is the total distance traveled by the in-line skater? [34 m]

10. The driver of a car going 90 km/h suddenly sees the lights of a barrier 40 m ahead. It takes the driver 0.75 s to apply the brakes, and the average acceleration during braking is –10.0 m/s^2. Does he hit the light barrier? [Yes]

11. A car is initially moving at 6 m/s and with a uniform acceleration of 2.4 m/s^2. Find its velocity after 6 s and the distance travelled in that time. [20.4 m/s; 79.2 m]

12. Light from the Sun reaches the Earth in 8.3 min. The speed of light is $3 \cdot 10^8$ m/s. How far is Earth from the Sun? [$1.5 \cdot 10^{11}$ m]

13. A car is moving down a street at 55 km/h. A child suddenly runs into the street. If it takes the driver 0.75 s to react and apply the brakes, how many meters will the car have moved before it begins to slow down? [11 m]

14. An automobile accelerates from rest at 2 m/s^2 for 20 s. The speed is then held constant for 20 s, after which there is an acceleration of –3 m/s2 until the automobile stops. What is the total distance traveled? [1467 m]

15. A construction worker accidentally drops a brick from a high scaffold. What is the velocity of the brick after 4 s? How far does the brick fall during this time? [39 m/s; 78 m]

16. A student drops a ball from a window 3.5 m above the sidewalk. How fast is it moving when it hits the sidewalk? [8.3 m/s]

17. A ball is thrown upward with an initial velocity of 20 m/s. How long is the ball in the air? *b*) What is the greatest height reached by the ball? *c*) When is the ball 15 m above the ground? [4.08 s; 20.4 m; 0.991 s; 3.09 s]

18. An object is dropped from a height of 120 m. Find the distance it falls during its final second in the air. [43.6 m]

19. An object is dropped from a height. During the final second of its fall, it traverses a distance of 38 m. What was the height? [93.8 m]

20. A stone is thrown vertically from a cliff 200 m tall. During the last half second of its flight the stone travels a distance of 45 m. Find the initial velocity of the stone. [±68 m/s, the stone may be thrown either up or down]

21. A rock dropped from a cliff falls one-third of its total distance to the ground in the last second of its fall. How high is the cliff? [145.7 m]

22. Two stones are dropped from the edge of a 60-m cliff, the second stone 1.6 s after the first. How far below the cliff is the second stone when the separation between the two stones is 36 m? [10.9 m]

23. At $t = 0$, a stone is dropped from a cliff above a lake. 1.6 s later another stone is thrown downward from the same point with an initial speed of 32 m/s. Both stones hit the water at the same time. Find the height of the cliff. [27.6 m]

24. Acceleration due to gravity on Mars is about one-third that on Earth. Suppose you throw a ball upward with the same velocity on Mars as on Earth. How would the ball's maximum height compare to that on Earth? How would its flight time compare? [Three times higher; the flight time would be three times as long]

25. A weather balloon is floating at a constant height above Earth when it releases a pack of instruments. If the pack hits the ground with a velocity of 73.5 m/s, how far did the pack fall? How long did it take for the pack to fall? [276 m; 7.50 s]

26. A car's displacement is given by the following equation $x = 3t + t^4$, where x is in meters and t is in seconds. What is the acceleration of the car at t = 0.5 s? [3 m/s^2]

27. A particles starts from rest and moves along a line. The equation for the displacement of the particle is $x = t^2 - 4t$, where x is in meters and t is in seconds. When does the particle stop? What type of motion is this? [2 s; motion with constant acceleration]

28. A particles starts from rest and moves along a line. The equation for the displacement of the particle is $x = t^3 - 4t^2$, where x is in meters and t is in seconds. When does the particle have no acceleration? What is the velocity of the particle at this time? [1.33 s; – 5.33 m/s]

29. A stone is dropped from the top of a cliff and take 6.3 s to reach the bottom. How high is the cliff? [98 m]

30. A block of wood slides off a horizontal table that is 1 m high. If the block of wood was traveling at 5 m/s when it left the table, how far from the edge of the table will the block land? [2.24 m]

31. A block of wood slides along a desk that is 0.8 m high. How fast should the block be moving when it leaves the desk so that it lands 3 m from the edge of the desk? [7.5 m]

32. A ball is kicked with a velocity of 12 m/s at an angle of 42° above the horizontal. How far away does the ball land? [14.3 m]

33. A ball is kicked off the edge of a 70 m cliff with a velocity of 16 m/s at an angle of 50° above the horizontal. How far from the edge of the cliff does the ball land? [53 m]

34. A stone thrown horizontally from the top of a 24-m tower hits the ground at a point 18 m from the base of the tower. (Ignore any effects due to air resistance.) Find the speed with which the stone was thrown and the speed of the stone just before it hits the ground. [8.1 m/s; 23 m/s]

35. A ball launched from ground level lands 2.44 s later 40 meters away from the launch point. Find the magnitude of the initial velocity vector and the angle it is above the horizontal. [20.3 m/s, 36.1°]

36. The football was punted and left the punter's foot at a height 1 m above the ground. How far did the football travel before hitting the ground? [40.5 m]

37. A cork shoots out of a champagne bottle at an angle of 35° above the horizontal. If the cork travels a horizontal distance of 1.3 m in 1.25 s, what was its initial speed? [1.27 m/s]

38. In a game of basketball, a forward makes a bounce pass to the center. The ball is thrown with an initial speed of 4.3 m/s at an angle of 15° below the horizontal. It is released 0.8 m above the floor. What horizontal distance does the ball cover before bouncing? [1.27 m]

39. Repeat the previous problem for a bounce pass in which the ball is thrown 15° above the horizontal. [2.21 m]

40. Snowballs are thrown with a speed of 13 m/s from a roof 7 m above the ground. Snowball A is thrown straight downward; snowball B is thrown in a direction 25° above the horizontal. When the snowballs land, is the speed of A greater than, less than, or the same as the speed of B? Explain. Verify your answer by calculating the landing speed of both snowballs. [18 m/s]

41. A merry-go-round rotates at a steady rate and completes one revolution in 15 s. Find the linear speed of a child standing on the merry-go-round *a*) 5 m from the center; *b*) 10 m from the center. [2.09 m/s; 4.18 m/s]

42. A compact disk starts from rest and accelerates to its final angular velocity of 3.5 rev/s in 1.5 s. Find the disk's average angular acceleration in revolutions per second squared and in radians per second squared. [14.7 rad/s^2]

43. A motor running at 2600 rev/min is suddenly switched off and decelerates uniformly to rest after 10 s. Find the angular deceleration and the number of rotations to come to rest. [27.23 rad/s^2; 1361 rad; 216.7 rev]

44. A car is to be driven at a steady speed of 80 km/h round a smooth bend which has a radius of 30 m. Calculate the linear acceleration towards the centre of the bend experienced by a passenger in the car. [16.5 m/s^2]

45. A cyclist negotiates the curvature of 20 m with a speed of 20 m/s. What is the magnitude of his acceleration? [20 m/s^2; centripetal]

46. An astronaut, executing uniform circular motion in a centrifuge of radius 10 m, is subjected to a radial acceleration of $4g$ ($g = 9.8$ m/s^2). Find the time period of the centrifuge. [3.14 s.]

47. A wheel is making revolutions about its axis with uniform angular acceleration. Starting from rest it reaches 100 rev/s in 4 s. Find the angular acceleration and the angle of rotation during these four seconds. [25 rev/s^2, 400π]

48. A wheel with a uniform angular acceleration covers 50 rev in the first five seconds after the start. Find the angular acceleration, the angular velocity at the end of 5 s. [25.1 rad/s^2; 12.6 rad/s]

49. A body rotates about a fixed axis with angular acceleration of 1 rad/s^2. Through what angle does it rotate during the time in which its angular velocity increases from 5 rad/s to 15 rad/s? [100 rad]

50. Find the angular velocity of a body rotating with an angular acceleration of 2 rev/s^2 as it completes the fifth revolution after the start. [4.47 rad/s]

Chapter 2. DYNAMICS

2.1. NEWTON'S LAWS OF MOTION

2.1.1. The basic quantities of dynamics

The mathematical description of motion that includes the quantities that affect motion – mass and force – is called dynamics. ***Dynamics*** is the study of *why* things move as they do.

Mass is the quantitative measure of inertia of a body. Objects change their motion in response to actions from external objects, and mass is the amount of opposition to changes in motion that an object possesses.

Inertia of a body is its reluctance to start moving, and its reluctance to stop once it has begun moving. Thus an object at rest begins to move only when it is pushed or pulled, i. e., when force acts on it. In general, inertia is resistance to changing. In mechanics, inertia is the resistance to changing in velocity or, if you prefer, the resistance to acceleration.

Mass is a scalar quantity associated with matter. When a system is composed of several objects it is the total mass that matters.

$[m]$ = kilogram = kg.

Force is a physical quantity that can affect the motion of an object. In mechanics, a force is an interaction that causes a change in velocity or an interaction that causes acceleration. Force is a vector quantity associated with an interaction. Since force is a vector quantity we use geometry instead of arithmetic when combining forces. When several forces act on a system it is the *net* (*total*) external force that matters.

$[F]$ = Newton = N = kg·m·s^{-2}.

Linear momentum (or *momentum*) of material objects is a vector quantity equal to the product of the object's mass and its velocity vector

$$\vec{p} = m \cdot \vec{v}\,. \tag{2.1}$$

$[p]$ = kg·m·s^{-1}.

2.1.2. Inertial frames of reference. Galileo's principle of relativity

A lot of reference frames may be used to describe the motion. Among them are the frames the motion respectively which is described most simply. A frame of reference that remains at rest or moves with constant velocity with respect to other frames of reference is called ***inertial frames of reference***. An inertial frame of reference is actually an unaccelerated frame of reference. Newton's laws of motion are valid in all inertial frames of reference. In this frame of reference a body does not acted upon by external forces. All inertial frames of reference are equivalent for the measurement of physical phenomena. An example of the inertial frame of reference is heliocentric frame (the origin of coordinates is the Sun and the axes are directed towards the distant stars).

The great Italian physicist Galileo Galilei (1564–1642) in his book "Dialogues concerning the two chief world systems) in 1632 formulated the principle that was later named after him.

Galileo's principle of relativity: The mechanical laws of physics are the same for every inertial observer. By observing the outcome of mechanical experiments,

one cannot distinguish a state of rest from a state of constant velocity.

Such an observer who moves uniformly with constant speed in a straight line, i. e., moves with constant velocity", is a special type of observer called an "inertial" observer. From now on, we use "inertial" instead of "moves with constant velocity". The Galilean principle of relativity abolished the universality of the notion of "an observer at rest". As far as no mechanical experiment can ever distinguish "rest" from "uniform velocity", thus, "absolute rest" has no universal meaning according to Galileo. The motion and rest are relative to the frame of reference.

2.1.3. Newton's laws of motion

Sir Isaac Newton (1642–1727) outstanding English physicist and mathematician in his book "Mathematical Principles of Natural Philosophy" (1687) defined three laws concerning the behavior of moving objects. These scientific statements help to explain the nature of matter and space. Newton's first law of motion is often called the *Law of Inertia*. His second law called the *Force Law* shows the relationship between force and acceleration. His third law is often called the *Action-Reaction Law of Motion* or the *Law of Reciprocal*

Action. These laws can be verified in many common experiments, and they explain how and why objects move when forces are applied to them.

The First Law: An object at rest tends to remain at rest and an object in motion tends to continue moving with constant velocity unless compelled by a net external force to act otherwise.

The Second Law: The change of momentum per second is proportional to the applied force and the momentum change takes place in the direction of the force.

$$\frac{d\vec{p}}{dt} = \vec{F}, \tag{2.2}$$

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a}. \tag{2.3}$$

The Third Law*:* Action and reaction are always equal and opposite.

$$\vec{F}_{12} = -\vec{F}_{21}. \tag{2.4}$$

It is necessary to note that these forces do not counterbalance each other since they are applied to different objects.

Note that all Newton's laws are valid only in inertial frames of reference.

2.2. FORCES IN DYNAMICS

2.2.1. Types of forces

All the forces can be explained in terms of four fundamental interactions.

1. The ***strong interaction*** is very strong, but very short-ranged. It acts only over ranges of order 10^{-15} meters and is responsible for holding the nuclei of atoms together. It is basically attractive, but can be effectively repulsive in some circumstances.

2. The ***electromagnetic force*** causes electric and magnetic effects. It is long-ranged, but much weaker than the strong force. It can be attractive or repulsive, and acts only between pieces of matter carrying electrical charge.

3. The ***weak force*** is responsible for radioactive decay and neutrino interactions. It has a very short range and, as its name indicates, it is very weak.

4. The ***gravitational force*** is weak, but very long ranged. It is always attractive, and acts between any two masses in the Universe since mass is its source.

A force is a push or pull acting upon an object as a result of its interaction with another object. A variety of force types were placed into two broad category headings on the basis of whether the force resulted from the contact or non-contact of the two interacting objects. Many forces are ***contact forces***; they act only while two objects are in physical contact. The examples of contact forces: friction force, tension, normal force, spring force, air resistance, applied force. Other forces (gravitational, electrical, and magnetic forces) are ***action-at-a distance*** forces for which no physical contact is necessary. These forces are closely connected to the concept of the field and for this reason have another name 'field forces'. In physics, ***field*** is the region throughout which a force may be exerted. Examples are the gravitational, electric, and magnetic fields that surround, respectively, masses, electric charges, and magnets. Fields are used to describe all cases where two bodies separated in space exert a force on each other. Each type of force (electric, magnetic, nuclear, or gravitational) has its own appropriate field; a body experiences the force due to a given field only if the body itself is also a source of that kind of field.

2.2.2. Gravitational interaction, gravity, reactions and weight

The Universal Law of Gravitation (*Newton's Law of Gravity*) states that every object in the universe attracts every other object in the universe with the gravitational force. The magnitude of the gravitational force between two objects of masses m_1 and m_2 is directly proportional to the product of their masses and inversely proportional to the square of the distance r between their centers.

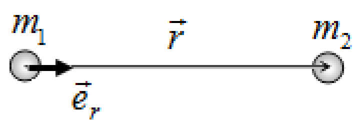


Figure 2.1

$$\vec{F} = \gamma \frac{m_1 \cdot m_2}{r^2} \cdot \vec{e}_r, \tag{2.5}$$

where $\gamma = 6.67 \cdot 10^{-11}$ m$^3 \cdot$s$^{-3} \cdot$kg^{-1} is the *gravitational constant*, $\vec{e}_r$ is a unit vector directed along a vector $\vec{r}$ (fig. 2.1).

Free fall occurs whenever an object is acted upon by gravity only.

$$m\vec{a} = \gamma \frac{m \cdot M}{R^2} \cdot \vec{e}_R = m\vec{g}\,, \tag{2.6}$$

where M and R are the mass and the radius of the Earth, respectively.

The force $m\vec{g}$ is **gravitational force** or **gravity**.

An object in free fall experiences the ***acceleration due to gravity*** $\vec{g}$.

$$\vec{a} = \vec{g} = \gamma \frac{M}{R^2} \cdot \vec{e}_R\,. \tag{2.7}$$

The acceleration due to gravity is independent on the mass of falling object.

The acceleration due to gravity varies with location. On the Earth this value varies with latitude and altitude. The acceleration due to gravity is greater at the poles than at the equator and greater at sea level than atop Mount Everest. There are also local variations that depend upon geology. For example, accelerations due to gravity are 9.8127 m/s^2 in Ukraine, 9.7952 m/s^2 in Iraq, 9.7921 m/s^2 in Lebanon, and 9.7755 m/s^2 in Nigeria. The value of 9.8 m/s^2 obtained through (2.8) is thus merely a convenient average over the entire surface of the Earth.

$$g = \gamma \frac{M}{R^2} = \frac{6.67 \cdot 10^{11} \cdot 5.97 \cdot 10^{24}}{\left(6.378 \cdot 10^6\right)^2} = 9.788 \approx 9.8 \text{ m/s}^2, \tag{2.8}$$

where $M = 5.97 \cdot 10^{24}$ kg is the Earth's mass, and $R = 6.378 \cdot 10^6$ m is an average Earth's radius.

If a body is suspended (1) or put on a base (2) the *gravity* $m\vec{g}$ is counterbalanced by force $\vec{R}$ (*reaction force*).

1) ***Tension*** $\vec{T}$ (fig. 2.2, *a*). This reaction force is the force of a string or rope on an object to which the string or rope is attached. The direction of the tension is always along the rope or string and *away* from the surface of the object to which the rope or string is attached. Tension forces can only *pull* the objects they act on. It is important to remember that we always assume a non–stretching string or rope (unless explicitly told otherwise) so that the magnitude of the tension is constant along the string or rope. This enormously simplifies the mathematics of using the tension by adding important constraints to the solution of the problem. One of the most important constraints is that the length of the string or rope is constant.

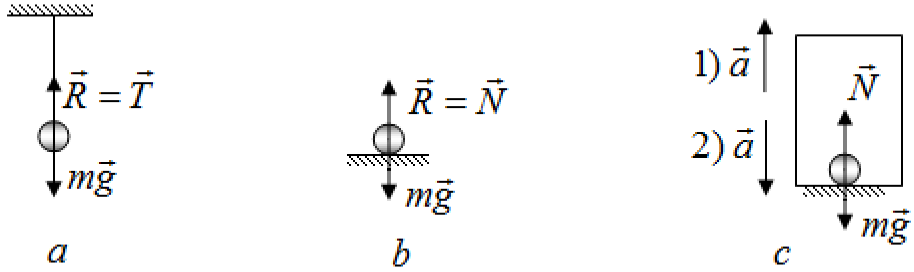


Figure 2.2

2) ***Normal force*** $\vec{N}$ (fig. 2.2, *b*). These are the reaction forces that result from contact between two objects. Normal forces are always directed perpendicularly *away* from the surface which exerts the normal force. Their magnitude depends on some external agent which maintains the contact between objects. To determine their direction for any situation, note that a normal force can only *push* the object it acts on.

The ***weight*** $\vec{G}$ of an object is defined as the force acting on it due to gravitational pull, or gravity. So the weight of an object can be measured by attaching it to a spring-balance or putting it on the scale. The reading of the scale is actually the normal force $\vec{N}$ that scale exerts back towards the object. At rest, $\vec{G}=-\vec{N}$, and $\vec{N}=-m\vec{g}$, as a result, the weight is equal to the gravitational force $\vec{G}=m\vec{g}$. The magnitudes of the weight, gravitational force and normal force are equal in value, $G=N=mg$, if the object is at rest or it is moving uniformly in the upward and downward direction.

Things get complicated when the scale or balance experience acceleration (fig. 2.2, *c*). The equation of the object motion in this case is

$$m\vec{a}=m\vec{g}+\vec{N}. \quad \vec{N}=m(\vec{a}-\vec{g}). \tag{2.9}$$

1) if the object is moving with upward acceleration $\vec{a}$, vectors $\vec{g}$ and $\vec{a}$ are of the opposite directions. This is the o*verload* when the weight G is greater than the gravity mg

$$G=N=m\left(g-(-a)\right)=m\left(g+a\right); \tag{2.10}$$

2) if the object is moving with downward acceleration $\vec{a}$, the vectors $\vec{g}$ and $\vec{a}$ are of the same direction. This is the *weightlessness* when the weight is less than the gravity

$$G=N=m\left(g-a\right). \tag{2.11}$$

When the downward acceleration of the object is equal to the acceleration due to the gravity $\vec{a} = \vec{g}$, the weight $G = 0$.

2.2.3. Spring forces. Hooke's law

A change in shape due to the application of a force is a ***deformation.*** Even very small forces are known to cause some deformation. The deformation is elastic if the object returns to its original shape when the force is removed. Elastic deformation is recoverable deformation. Plastic deformation is a process in which enough stress is placed on object to cause it to change its size or shape in a way that is not reversible.

Robert Hook (1635–1703) observed that when an elastic body is subjected to stress its dimension or shape changes in proportion to the applied stress over a range of stresses. The force F arising in an object under deformation (typically extension or compression) and returning it to its original shape when released (like a spring or elastic band) is the ***spring*** (or ***restoring***) ***force***. The relationship among force and deformation is called ***Hook's Law***.

$$F = -kx. \tag{2.12}$$

Let x denote the displacement from equilibrium for spring (elastic) system (fig. 2.3). This elongation (or shortening) of the object is proportional to the magnitude of the external force.

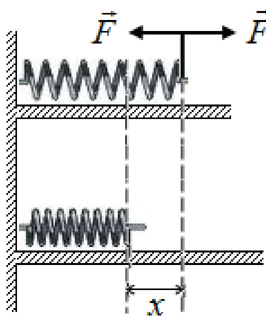


Figure 2.3

$$x \sim F_{\text{ext}}.$$

According to the Newton's 3rd Law external force F_{ext} and spring (elastic) force F are equal in value and opposite in sign $F_{\text{ext}} = -F$. The elastic force is greater the greater the elongation of the spring (or any elastic object) due to the external force acting, therefore, $F \sim x$, or

$$F = -kx,$$

where F is the spring force, k is the ***spring constant***, and x is the amount by which the spring is stretched ($x > 0$) or compressed ($x < 0$). The minus sign indicates that the direction of the force F is *opposite* to the displacement of the spring.

In 1678, Robert Hooke, on the basis of his experiments with springs, stretching wires, and coils, stated a rule that relates the stress of a body to the strain in the elastic range. Strain ε is a term used to measure the deformation or extension of a body that is subjected to a force or set of forces. The ***strain*** of a body is generally defined as the change in length divided by the initial length $\varepsilon = \Delta x / x$. The ***stress*** is the ratio of the applied force and cross-sectional area normal to this force $\sigma = F / S_{\perp}$. The ***stress-strain relationship*** is

$$\sigma = E\varepsilon, \tag{2.13}$$

where E is the material property that represents the stiffness of the material called ***Young's Modulus***. Young's moduli are determined through experiments and are commonly listed in engineering handbooks.

2.2.4. Friction. Types of friction. Amontons-Coulomb's laws

Friction is the force resisting the relative motion of solid surfaces, liquid and gas layers, and material elements sliding against each other. Friction is directed opposite to the direction of the relative motion or the intended direction of motion of either of the surfaces.

Types of friction depend on different principals of their classifying.

There are dry and fluid frictions, and intermediate type – friction with lubrication.

Friction may be

- *internal* (or viscosity) – arises on account of relative motion between every two layers of a liquid or gas;
- *external* (or contact) – arises when two bodies in contact with each other try to move or there is an actual relative motion between the two.

External friction is of three types

- *static* friction – results when the surfaces of two objects are at rest relative to one another and a force exists on one of the objects to set it into motion relative to the other object but the actual motion has yet not started. The magnitude of the static friction is not constant. It always adjusts itself so as to be equal to the applied force.
- *limiting* friction – when the body is just at the verge of the moving over the other. The static friction at this stage is obviously maximum;

- *kinetic* (or dynamic) friction – that takes place during motion and may be divided by *sliding* friction and *rolling* friction.

It was Guillaume Amontons (1663–1705) who first established a proportional relationship between friction force and the mutual pressure (or normal force) between the bodies in contact. An empirical law for the friction between two sliding solids was published by Charles Augustin de Coulomb (1736–1806).

Amontons-Coulomb's laws for friction are

1. The friction force is independent of the apparent contact area between the sliding surfaces.
2. The friction force is proportional to the *normal force* N. The proportionality factor is called ***coefficient of friction***.
3. The kinetic friction force $\left(F_{fr}\right)_K$ that keeps a body sliding at a constant velocity does not depend on the sliding velocity. It is less than or equal to the **static friction** force $\left(F_{fr}\right)_S$, i. e., the force to start sliding.

$$\left(F_{fr}\right)_K = \mu_K N, \quad \left(F_{fr}\right)_S = \mu_S N, \quad \mu_K \le \mu_S , \tag{2.14}$$

where μ_K and μ_S are the coefficients of kinetic and static friction, respectively, N is the normal force pressing the surfaces together.

In physics, liquids and gases are both called ***fluids***. The friction that happens with liquids and gases is ***fluid*** (*internal*) ***friction***. It depends on how thick fluid is (on its viscosity), on the shape of the object and the velocity of the object.

Drag forces are forces an object experiences opposing their motion in a fluid. Air is considered a fluid and so air resistance is normally considered as drag force.

Small objects moving at *slow speeds* can be treated as having a drag force (***Stokes' Law***)

$$F_d = -bv, \tag{2.15}$$

where v is the speed of the moving object, and b is the parameter depending on the nature of the fluid and dimensions of the object.

For example, parameter b for the spherical object moving in the fluid is

$$b = 6\pi\mu_f R\,, \tag{2.16}$$

where μ_f is the *dynamic viscosity* of the fluid and R is the radius of the spherical object.

Drag at *high velocities* depends as

$$F_D = \frac{\rho \cdot C_D \cdot S}{2} v^2\,, \tag{2.17}$$

where ρ is the mass density of the fluid, C_D is the drag coefficient – a dimensionless coefficient related to the object's geometry, S is the reference area (the orthographic projection of the object on a plane perpendicular to the direction of motion), v is the velocity of the object relative to the fluid.

When an object falls, it accelerates. As its speed increases, the air resistance increases. Eventually, the force from the air resistance will equal the force from the weight of the object. At that point, the speed will remain constant: the object has reached its "*terminal velocity*" and can't fall any faster.

$$v_{\mathrm{t}} = \sqrt{\frac{2mg}{C_D \rho S}}\,. \tag{2.18}$$

Terminal velocity depends on the drag, so a streamlined shape will fall quickly, whilst a parachute will fall slowly.

Frictions can produce both positive and negative effects. The useful action of the friction appears during walking, car braking, some technological techniques like friction welding, polishing and other types of treatment, etc. On other side, friction can cause physical wear and tear of moving parts, degradation of materials by continuous friction, wastage of energy, etc. The friction can be reduced by lubricating the surfaces by different oils and greases, by changing the type of friction (rolling instead of sliding), and usage of special materials with low coefficients of friction (ceramics, Teflon, etc).

2.3. THE CONSERVATION OF LINEAR MOMENTUM

2.3. 1. System in physics. The law of conservation of linear momentum

A system is a portion of the universe (the material objects) that has been chosen for studying the changes that take place within it in response to varying conditions. These objects may interact with each other with internal forces and with the objects outside the system with external forces.

Closed **(*isolated*)** ***system*** is the system on which no external forces act.

Let the system consist of N particles. $\vec{F}_{ik}$ is the internal force with which k-particle acts on i-particle, $\vec{F}_i$ is the net external force acting on i-particle.

The equations of motion for all particles of the system are:

$$\begin{cases} \dfrac{d\vec{p}_1}{dt} = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14} + ... + \vec{F}_{1N} + \vec{F}_1, \\ \dfrac{d\vec{p}_2}{dt} = \vec{F}_{21} + \vec{F}_{23} + \vec{F}_{24} + ... + \vec{F}_{2N} + \vec{F}_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dfrac{d\vec{p}_N}{dt} = \vec{F}_{N1} + \vec{F}_{N2} + \vec{F}_{N3} + ... + \vec{F}_{N,N-1} + \vec{F}_N. \end{cases} \tag{2.19}$$

Add the left and right sides of above equations separately. The sum of the left-hand members is

$$\frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} + ... + \frac{d\vec{p}_N}{dt} = \frac{d(\vec{p}_1 + \vec{p}_2 + ... + \vec{p}_N)}{dt} = \frac{d\vec{p}}{dt}, \tag{2.20}$$

where $\vec{p}$ is the *total linear momentum* of the system.

The sum of right-hand members is

$$(\vec{F}_{12} + \vec{F}_{21}) + (\vec{F}_{13} + \vec{F}_{31}) + ... + (\vec{F}_{1N} + \vec{F}_{N1}) + \sum_{i=1}^{N} \vec{F}_i = \sum_{i=1}^{N} \vec{F}_i, \tag{2.21}$$

since all sums in the brackets are equal to zero according to the Newton's 3rd law. As a result,

$$\frac{d\vec{p}}{dt} = \sum_{i=1}^{N} \vec{F}_i. \tag{2.22}$$

The rate of change of ***total linear momentum*** of the system is equal to the net force acting on it, and is pointed in the direction of the force.

For closed system, the net force on the system is zero $\sum_{i=1}^{N} \vec{F}_i = 0$; therefore, the linear momentum is constant.

$$\vec{p} = const. \tag{2.23}$$

This is the mathematical expression for the ***law of conservation of linear momentum***: the linear momentum of any closed system remains a constant quantity.

2.3.2. Impulse and average force. Impulse-momentum theorem

The expression for the rate of change of total linear momentum

$$\frac{d\vec{p}}{dt} = \sum_{i=1}^{N} \vec{F}_i = \vec{F}_{\text{net}}$$

gives

$$d\vec{p} = \vec{F}_{\text{net}} dt.$$

Integrating both sides, we find for the net change in momentum

$$\Delta\vec{p} = \int_0^{t_1} \vec{F}_{\text{net}} dt. \tag{2.24}$$

This integral (over time) of the force is called the ***impulse***.

Impulse-momentum theorem: The impulse of the net force is equal to the change of the total linear momentum.

One use of this relation is to define the average force that acts during a specified time interval. Let the force act for time Δt, producing a net change $\Delta\vec{p}$ in the total momentum.

The average force is given by

$$\vec{F}_{\text{ave}} = \frac{\Delta\vec{p}}{\Delta t}. \tag{2.25}$$

2.3.3. Center of mass of mechanical system and the theorem of its motion

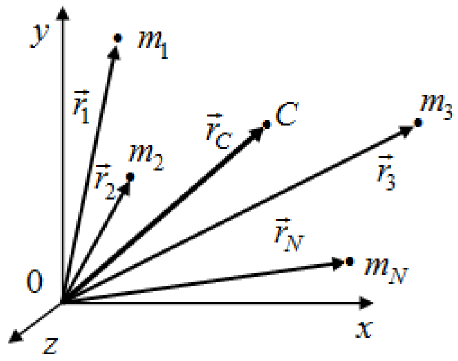


Figure 2.4

Consider a system of consisting of N iscrete particles each of mass m_i, which location is determined by positional vector $\vec{r}_i$. The position of ***the center of mass*** is defined by vector $\vec{r}_C$:

$$\vec{r}_C = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots + m_N\vec{r}_N}{m_1 + m_2 + \ldots + m_N} = \frac{1}{m}\sum_{i=1}^{N} m_i\vec{r}_i\,, \tag{2.26}$$

where m is the total mass of the system.

The rigid body can be treated as a continuous distribution of matter consisting of the differential mass elements dm. The vector position of the center of mass of the extended object is

$$\vec{r}_C = \lim_{\Delta m_i \to 0}\left(\frac{1}{m}\sum_{i=1}^{N} m_i\vec{r}_i\right) = \frac{1}{m}\int \vec{r}dm\,. \tag{2.27}$$

Taking the derivative of (2.26) with respect to time, we obtain

$$\frac{d\vec{r}_C}{dt} = \vec{v}_C = \frac{1}{m}\sum_{i=1}^{N} m_i\frac{\vec{r}_i}{dt} = \frac{1}{m}\sum_{i=1}^{N} m_i\vec{v}_i = \frac{1}{m}\sum_{i=1}^{N}\vec{p}_i = \frac{\vec{p}}{m}\,.$$

Momentum of the system is equal to the product of the system total mass and the velocity of its center of mass

$$\vec{p} = m\vec{v}_C\,. \tag{2.28}$$

Differentiation of (2.28) with respect to time gives

$$\frac{d\vec{p}}{dt} = \frac{d(m\vec{v}_C)}{dt} = m\frac{d\vec{v}_C}{dt} = m\vec{a}_C = \sum_{i=1}^{N}\vec{F}_i\,. \tag{2.29}$$

This equation shows that the motion of the center of mass is determined only by the external forces. The sum of all internal forces (i. e., the forces exerted by one part of the system on its other parts) cancels out according to Newton's third law (since they are equal in magnitude but pointing in opposite direction).

The ***theorem of a center of mass motion***: The overall motion of a system of particles can be found by applying Newton's Laws as if the total mass of the system were concentrated at the center of mass and the external forces were applied at this point.

Hence, we can describe the translational motion of a rigid body as if it is a point particle with the total mass located at the center of mass.

$$m\vec{a}_C = \sum_{i=1}^{N} \vec{F}_i . \qquad (2.30)$$

Equation (2.30) shows that when the net external force acting on the system is zero, i. e., the system is closed, $\sum_{i=1}^{N} \vec{F}_i = 0$, and

$$m\vec{a}_C = 0, \qquad (2.31)$$

In other words, the center of mass of a closed system moves uniformly along a straight line or stays at rest. Therefore, it is convenient to choose it as the origin of inertial frame of reference.

2.4. THE CONSERVATION OF ENERGY

2.4.1. Work. Power

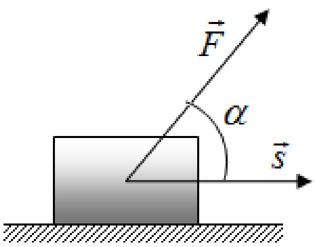


Figure 2.5

Let's assume that the constant force $\vec{F}$ acts on the object during its motion (fig. 2.5). Both the force and displacement are vectors that are not necessarily pointing in the same direction. The elementary ***work*** done by the constant force $\vec{F}$ as it undergoes a displacement $d\vec{s}$ is defined as

$$dA = (\vec{F}, d\vec{s}) = F \cdot ds \cdot \cos\alpha = F_s ds . \qquad (2.32)$$

where ds is a magnitude of displacement, and $F_s = F\cos\alpha$ is the projection of the force on the direction of displacement.

The total work of constant force is

$$A = \left(\vec{F}, \vec{s}\right) = F \cdot s \cdot \cos\alpha = F_s \cdot s , \qquad (2.33)$$

The work done by the force $\vec{F}$ is zero if the displacement is equal to zero or if $\alpha = 90°$, i. e., the force is perpendicular to the displacement.

According to definition (2.32), the work is a scalar.

The work done by the force can be positive or negative depending on α. If $\alpha < 90°$, the work is positive; if $\alpha > 90°$, the work is negative.

When a system does work on its environment, $A > 0$; that is, the total energy of the system *decreases*. Work is done *by* the system.

When the environment does work on a system, $A < 0$; that is, the total energy of the system *increases*. Work is done *on* the system.

When a varying force $\vec{F} = \vec{F}(\vec{s})$ is acting on an object, the work is

$$A = \int_1^2 dA = \int_1^2 (\vec{F}, d\vec{s}) = \int_1^2 F_s ds \,. \qquad (2.24)$$

Work done by the constant force for displacement $s = s_2 - s_1$ equals the area of the shaded rectangle (fig. 2.6, *a*). The work done by the component F_s the varying force F as the object moves from s_1 to s_2 is exactly equal to the area under this curve (fig. 2.6, *b*).

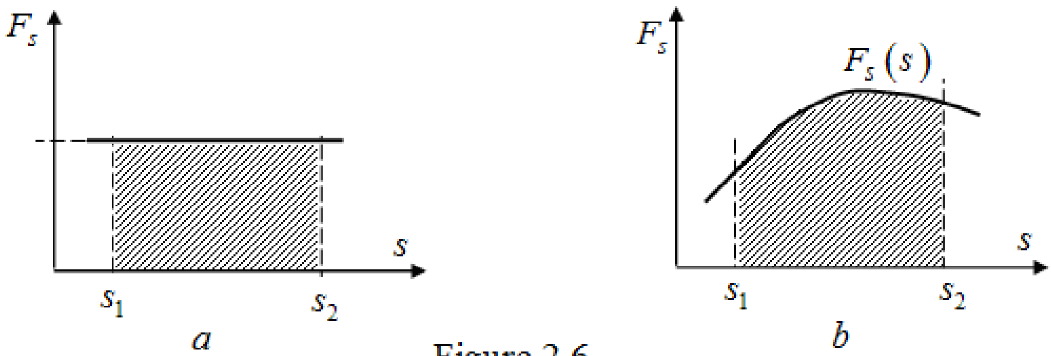


Figure 2.6

If more than one force acts on the object, the total work done is just the work done by the resultant (net) force $\vec{F}$.

$$A = (\vec{F}, d\vec{s}) = (\sum_{i=1}^{N} \vec{F}_i, d\vec{s}) = (\vec{F}_1, d\vec{s}) + \ldots + (\vec{F}_N, d\vec{s}) = A_1 + A_2 + \ldots A_N \,. \qquad (2.35)$$

$[A]$ = Joule = J = kg·m^2·s^{-2}.

P***ower*** is the time rate of doing work. If an external force is applied to an object, and if the work done by this force in the time interval Δt is A then the **average power** expended during this interval is defined as

$$P_{\text{ave}} = \frac{A}{\Delta t}. \tag{2.36}$$

The work done on the object contributes to the increase in the energy of the object. Therefore, ***power*** maybe defined more generally as time rate of energy transfer.

The instantaneous power is the limiting value of the average power as Δt approaches zero.

$$P = \lim_{\Delta t \to 0} \frac{A}{\Delta t} = \frac{dA}{dt}, \tag{2.37}$$

where dA is the increment of work done.

Using (2.32) gives

$$P = \frac{dA}{dt} = \frac{(\vec{F}, d\vec{s})}{dt} = (\vec{F}, \frac{d\vec{s}}{dt}) = (\vec{F}, \vec{v}). \tag{2.38}$$

The SI unit for power is Watt

$[P]$ = Watt = W = J/s = kg·m^2·s^{-3}.

A unit of power in the engineering system is horsepower 1 hp = 746 W.

2.4.2. Energy. Kinetic energy

The work is closely related to energy. The work causes a change in energy and energy characterizes the ability to do the work, i. e., work is a quantitative measure of changes in energy.

In mechanics, it is possible to define energy as an ability to do work and to consider it as the maximum work which any object can do under the given conditions. But generally it is not so. For example, thermal energy cannot be transformed completely into the work. A system possesses energy if it has the ability to do work. Energy is a scalar quantity which is given meaning through calculation.

Energy is a fundamental entity of nature that is transferred between parts of a system in the production of physical change within the system and usually regarded as the capacity for doing work. Energy can exist in different forms.

If the force $\vec{F}$ acts on mass m changing its velocity from v_1 to v_2, the work done by this force is

$$dA = F_s ds = m \cdot \frac{dv}{dt} \cdot ds = m \cdot v \cdot dv. \quad (2.39)$$

The ***Work-Energy Theorem*** (*Work-Energy Principle*) states that the work causes a change in velocity and, as a result, a change in energy associated with velocity, therefore,

$$A = \int_1^2 dA = \int_{v_1}^{v_2} m \cdot v \cdot dv = m\int_{v_1}^{v_2} vdv = \frac{mv^2}{2}\bigg|_{v_1}^{v_2} = \frac{mv_2^2}{2} - \frac{mv_1^2}{2} = W_{k2} - W_{k1} = \Delta W_k. \quad (2.40)$$

We obtained the expression for energy associated with motion and called the ***kinetic energy****:*

$$W_k = \frac{mv^2}{2}. \quad (2.41)$$

Notice that the work is equal to the increment of the kinetic energy

$$A = W_{k2} - W_{k1} = \Delta W_k. \quad (2.42)$$

2.4.3. Potential energy

1. Let us find the work of gravity during the material point motion along a curvilinear path (acceleration due to gravity is nearly constant and height change is small compared with the distance between centers of the Earth and the particle) (fig. 2.7).

$$\Delta A = mg \cdot \Delta s_k \cos\alpha = mg\Delta h_k.$$

The work causes a change in energy

$$\begin{aligned} A &= \sum_k \Delta A_k = mg\sum_k \Delta h_k = mgh = mg(h_1 - h_2) = \\ &= mgh_1 - mgh_2 = W_{p1} - W_{p2} = -\Delta W_p. \end{aligned} \quad (2.43)$$

We obtained the expression for energy associated with position and called the *potential energy.*

The ***potential energy due to gravitational force*** is

$$W_p = mgh. \tag{2.44}$$

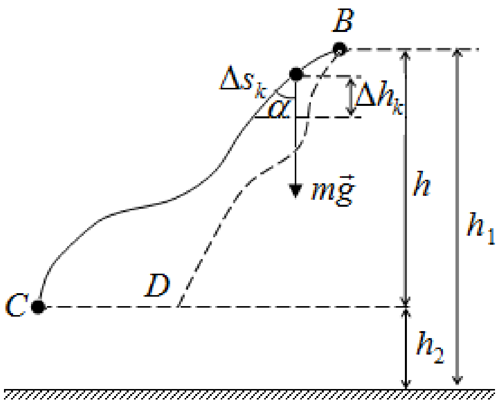


Figure 2.7

We see that in evaluating the work done by the force during the motion, no mention is made of the actual path taken by the particle. The work done by the gravitational force does not depend on whether an object falls vertically or, for example, slides down a sloping incline. All that matters is the change in the object's elevation Δh.

A force is ***conservative*** if the work it does on a particle that moves between two points is the same for all paths connecting these points; i. e., is independent of the path taken by the particle; otherwise, the force is ***non-conservative***.

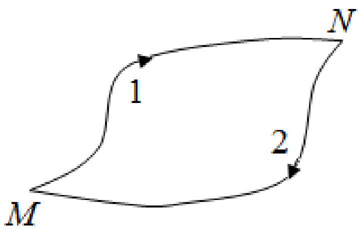


Figure 2.8

If a force is conservative, the *work* it does on a particle moving through any *closed path* is zero. A closed path is one in which the beginning and end points are identical (fig. 2.8).

Let's assume that the work done for the round trip from M to N and back to M is zero. This means that

$$A = \left(A_{MN}\right)_1 + \left(A_{NM}\right)_2 = 0. \tag{2.45}$$

The work done by the force on each segment reverses sign if we reverse the direction

$$\left(A_{NM}\right)_2 = -\left(A_{MN}\right)_2. \tag{2.46}$$

Since the work of conservative force does not depend on path and depends only on the positions of the initial and final points of motion

$$\left(A_{MN}\right)_2 = \left(A_{MN}\right)_1. \tag{2.47}$$

As a result of (2.46) and (2.47),

$$(A_{NM})_2 = -(A_{MN})_1 . \tag{2.48}$$

Accordingly (2.48),

$$A = (A_{MN})_1 - (A_{MN})_2 = (A_{MN})_1 - (A_{MN})_1 = 0 .$$

This is exactly which was to be proved.

2. The potential energy is closely connected to existence of field of forces (gravitational, electric). The field, in which a particle moves under the influence of a force that acts on the particle in such a way that it is always directed towards a single point (the center of force), is so-call ***central-force field***. The magnitude of any force depends on a distance from this center r (fig. 2.9). The force is directed either to the force center or from it. The examples are gravitational field of the Earth $F(r) = \gamma \frac{m_1 \cdot m_2}{r^2}$ and electric field of the point charge $F(r) = k\frac{q_1 \cdot q_2}{r^2}$.

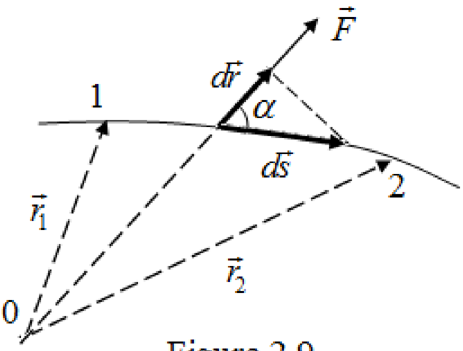


Figure 2.9

Let us find the work of the force $\vec{F}(r)$ that depends on the distance from the center r in the central-force stationary (independent on time) field. In equations

$$\begin{cases} F(r) = \gamma \dfrac{m_1 \cdot m_2}{r^2} \\ F(r) = k\dfrac{q_1 \cdot q_2}{r^2} \end{cases}$$

put

$$\beta = \begin{cases} \gamma \cdot m_1 \cdot m_2 \\ k \cdot q_1 \cdot q_2 \end{cases}, \tag{2.49}$$

and the both forces be as follows

$$F(r) = \frac{\beta}{r^2} . \tag{2.50}$$

The elementary work

$$dA = \vec{F}(r)d\vec{s} = F(r)ds \cdot \cos\alpha = F(r)dr = \frac{\beta}{r^2}dr\,. \qquad (2.51)$$

Integrating (2.51) along the path 1–2, we find the work done by force

$$A = \int_1^2 dA = \int_1^2 \frac{\beta}{r^2}dr = -\frac{\beta}{r}\bigg|_{r_1}^{r_2} = \frac{\beta}{r_1} - \frac{\beta}{r_2} = W_{p1} - W_{p2}\,. \qquad (2.52)$$

The ***potential energy of interaction*** (***mutual potential energy***) is inversely proportional to the distance between the objects.

Gravitational potential energy is

$$W_p = \gamma \frac{m_1 \cdot m_2}{r}\,. \qquad (2.53)$$

Electric potential energy is

$$W_p = k\frac{q_1 \cdot q_2}{r}\,. \qquad (2.54)$$

The potential energy is defined with the accuracy to constant of integrating as any integral. But the nature of potential energy is that the *zero point* is arbitrary; it can be set like the origin of a coordinate system. That is not to say that it is insignificant; once the zero of potential energy is set, then every value of potential energy is measured with respect to that zero. Another way of saying it is that it is the *change* in potential energy which has physical significance.

When we use the gravitational potential energy $W_p = mgh$, the assumption is usually made is that the zero of gravitational potential energy is on the surface of the Earth and that the potential energy is proportional to the height above the Earth's surface. This is an approximation which is only valid near the surface of the Earth, but it is suitable for the common applications of gravitational potential energy. If you are in a room, it is logical to just call the floor the zero of gravitational potential energy, and measure the energy of an elevated object with respect to the floor.

When you use the more general form of the gravitational potential energy, including the fact that it drops off with distance from the Earth, $F(r) = \gamma \dfrac{m_1 \cdot m_2}{r^2}$, then the logic of the choice of zero point is different. In this case, we generally choose the zero of gravitational potential energy at infinity, since the gravitational force approaches zero at infinity. This is a logical way to define the zero since the potential energy with respect to a point at infinity tells us the energy with which an object is bound to the Earth. (This more general case is similar to what is done with the zero of electrical potential, since it is logical to define the zero of voltage far away from any charges).

3. Let us calculate the ***potential energy due to elastic forces***.

The force exerted by a spring on a mass m can be calculated using Hooke's law (2.12) $F=-kx$. Since the elastic force $\vec{F}$ is collinear to the displacement $d\vec{x}$, $\cos\alpha=1$, and the elementary work is

$$dA=\left(\vec{F},d\vec{x}\right)=F\cdot dx\cdot\cos\alpha=F\cdot dx=-kx\cdot dx.$$

The work of the elastic force is

$$A=\int_1^2 dA=-\int_{x_1}^{x_2}kxdx=-\frac{kx^2}{2}\Bigg|_{x_1}^{x_2}=\frac{kx_1^2}{2}-\frac{kx_2^2}{2}=W_{p1}-W_{p2}. \tag{2.55}$$

The elastic potential energy is

$$W_p=\frac{kx^2}{2}. \tag{2.56}$$

The potential energy of the spring in its relaxed position is defined as zero.

2.4.4. Relationship between conservative forces and potential energy

In all examined cases of potential energy the work done by the conservative force equals the negative of the change in the potential energy associated with that force, that is,

$$A=W_{p1}-W_{p2}=-\left(W_{p2}-W_{p1}\right)=-\Delta W_p. \tag{2.57}$$

If the point of application of the force undergoes an infinitesimal displacement dx, we can express the infinitesimal change in the potential energy of the system as

$$dW_p=-F_xdx. \tag{2.58}$$

It follows that the conservative force is related to the potential energy function through the relationship

$$F_x = -\frac{dW_p}{dx}. \qquad (2.59)$$

That is, any conservative force acting on an object within a system equals the negative derivative of the potential energy of the system with respect to x.

2.4.5. The law of conservation of energy

The ***total mechanical energy*** W of a system is the sum of the energy associated with motion and the energy associated with position, i. e., the sum of its kinetic and potential energies:

$$W = W_k + W_p. \qquad (2.60)$$

Let us consider the system consisting of particles exerting forces on each other and estimate the work done at the displacement from one position to another, accompanied by the modification of configuration of the system.

The work of external conservative forces is

$$A'_{12} = -\Delta W'_p = -\left(W'_{p2} - W'_{p1}\right) = W'_{p1} - W'_{p2}. \qquad (2.61)$$

The work of internal conservative forces is

$$A''_{12} = -\Delta W''_p = -\left(W''_{p2} - W''_{p1}\right) = W''_{p1} - W''_{p2}. \qquad (2.62)$$

The work of non-conservative forces is A^*_{12}.

The total work of external and internal conservative and non-conservative forces is

$$A_{12} = A'_{12} + A''_{12} + A^*_{12}. \qquad (2.63)$$

The total work of all forces is expended on an increment in kinetic energy of a system:

$$A_{12} = \Delta W_k = W_{k2} - W_{k1}. \qquad (2.64)$$

Combining (2.63) and (2.64), we obtain

$$W_{k2} - W_{k1} = (W'_{p1} - W'_{p2}) + (W''_{p1} - W''_{p2}) + A^*_{12}. \tag{2.65}$$

Recombination of the members gives

$$\left(W_{k2} + W'_{p2} + W''_{p2}\right) - \left(W_{k1} + W'_{p1} + W''_{p1}\right) = A^*_{12}. \tag{2.66}$$

The sums in the brackets of equation (2.66) are the total energies in final and initial state, respectively. As a result, we obtain that the total energy of the system is changed due to the work A^*_{12} of the nonconservative forces

$$W_2 - W_1 = A^*_{12}. \tag{2.67}$$

Due to the absence of the nonconservative forces in the isolated (closed) system $A^*_{12} = 0$, and $W_2 - W_1 = 0$. As a result $W_2 = W_1$ or

$$W = \text{const}. \tag{2.68}$$

The law of conservation of energy: The total mechanical energy of a system remains constant in any isolated system of objects that interact only through conservative forces.

In other words: In an isolated system where only conservative forces cause energy changes, the kinetic energy and potential energy can change, but their sum, the mechanical energy W of the system, cannot change.

In the presence of nonconservative forces, mechanical energy is converted into internal energy or thermal energy.

2.4.6. Elastic and inelastic collisions

Collision is an event during which two objects come close to each other and interact by means of forces. A collision between two particles is said to occur if they physically strike against each other or if the path of motion of one is influenced by the other. In physics, the term collision does not necessarily mean that a particle actually strikes. In fact, two particles may not even touch each other and yet they are said to collide if one particle influences the motion of the other. Collisions are divided into two types: elastic collision and inelastic collision.

An ***elastic collision*** between two objects is one in which the total kinetic energy and the total momentum of the system is the same before and after the collision. An *inelastic collision* is one in which the total kinetic energy of the system is not the same before and after the collision, but the momentum of the system is conserved.

When the objects stick together after they collide, the collision is called ***perfectly inelastic.*** In this case two objects of masses m_1 and m_2 moving with velocities $\vec{v}_1$ and $\vec{v}_2$, respectively (fig. 2.10). They collide, stick together, and move with common velocity $\vec{u}$ after the collision. Equation according to the law of conservation of linear momentum for this system of two objects is

$$m_1\vec{v}_1 + m_2\vec{v}_2 = (m_1 + m_2)\vec{u}\,. \tag{2.69}$$

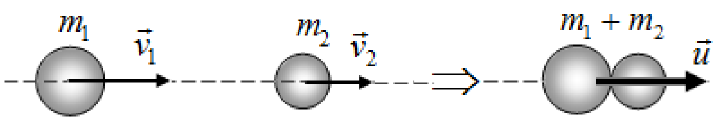


Figure 2.10

Solving for the final velocity gives

$$\vec{u} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}\,. \tag{2.70}$$

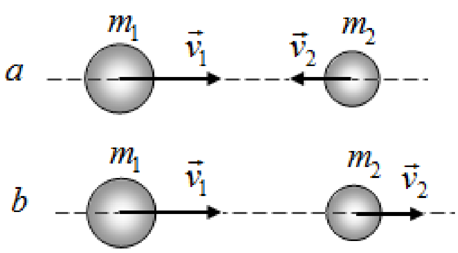


Figure 2.11

For the *elastic* collision, two objects of masses m_1 and m_2 moving with velocities $\vec{v}_1$ and $\vec{v}_2$, respectively, collide, and leave the collision site with velocities $\vec{u}_1$ and $\vec{u}_2$ (fig. 2.11). Since both the momentum and kinetic energy of the system are conserved in an elastic collision, we have

$$m_1\vec{v}_1 + m_2\vec{v}_2 = m_1\vec{u}_1 + m_2\vec{u}_2\,, \tag{2.71}$$

$$\frac{m_1\vec{v}_1^{\,2}}{2} + \frac{m_2\vec{v}_2^{\,2}}{2} = \frac{m_1\vec{u}_1^{\,2}}{2} + \frac{m_2\vec{u}_2^{\,2}}{2}\,. \tag{2.72}$$

Rewrite the equation (2.71) as

$$m_1(\vec{v}_1 - \vec{u}_1) = m_2(\vec{u}_2 - \vec{v}_2)\,, \tag{2.73}$$

and the equation (2.72) as

$$m_1\left(\vec{v}_1^2-\vec{u}_1^2\right)=m_2\left(\vec{u}_2^2-\vec{v}_2^2\right). \tag{2.74}$$

Factoring both sides of the equation (2.74) gives

$$m_1\left(\vec{v}_1-\vec{u}_1\right)\left(\vec{v}_1+\vec{u}_1\right)=m_2\left(\vec{u}_2-\vec{v}_2\right)\left(\vec{u}_2+\vec{v}_2\right). \tag{2.75}$$

Divide equation (2.75) by (2.73) and obtain

$$\vec{v}_1+\vec{u}_1=\vec{u}_2+\vec{v}_2\,. \tag{2.76}$$

Multiplication of (2.76) by m_2 and subtraction of the result from (2.73), and then multiplication of (2.76) by m_1 and addition of the result with (2.73) gives the velocities of the objects after the elastic collision

$$\begin{cases}\vec{u}_1=\dfrac{2m_2\vec{v}_2+\left(m_1-m_2\right)\vec{v}_1}{m_1+m_2},\\ \vec{u}_2=\dfrac{2m_1\vec{v}_1+\left(m_2-m_1\right)\vec{v}_2}{m_1+m_2}.\end{cases} \tag{2.77}$$

The projections on the $\vec{v}_1$-directions are

$$\begin{cases}u_1=\dfrac{\mp 2m_2v_2+\left(m_1-m_2\right)v_1}{m_1+m_2},\\ u_2=\dfrac{2m_1v_1\mp\left(m_2-m_1\right)v_2}{m_1+m_2},\end{cases} \tag{2.78}$$

where v_1 and v_2 are the magnitudes of the initial velocities; $u_1$and u_2 are the projections of the vectors on above mentioned direction. The sign “minus” relates to the case when the objects coming from opposite directions; the sign “plus” is for case when the first object overtakes the second object.

2.5. THE CONSERVATION OF ANGULAR MOMENTUM

2.5.1. Torque. Couple of forces

The force $\vec{F}$ acts on the particle of the rigid body pivoted about the point O which is called the *pivot point* (or *fulcrum*) and the particle's position relative to the point O is given by position vector $\vec{r}$ (fig. 2.12). The torque (***moment of a force***) acting on the particle is vector quantity defined as

$$\vec{M} = \left[\vec{r}, \vec{F}\right]. \tag{2.79}$$

The magnitude of the torque is

$$M = r \cdot F \cdot \sin\alpha = F \cdot l, \tag{2.80}$$

where α is the smaller angle between the directions of $\vec{r}$ and $\vec{F}$ when vectors are tail to tail.

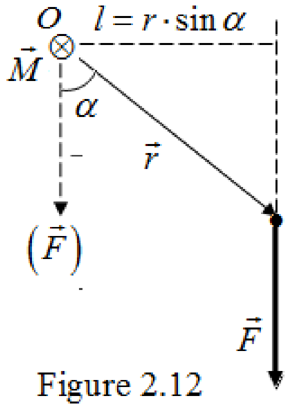


Figure 2.12

The torque shows the ability of the force $\vec{F}$ to rotate the body. This ability depends not only on the magnitude of the force but also on just how far from the point O the force is applied. An extended line running through the vector $\vec{F}$ is called the line of action of $\vec{F}$ and $l = r \cdot \sin\alpha$ is called the moment arm (the lever arm) of $\vec{F}$.

$$[M] = \mathrm{N} \cdot \mathrm{m} = \mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2}.$$

The SI unit of torque is the Newton·meter, which is also a way of expressing a Joule (the unit for energy). However, torque is *not* energy. So, to avoid confusion, we will use the units N·m, and not J. The distinction arises because energy is a scalar quantity, whereas torque is a vector.

The projection of torque vector to an arbitrary axis z containing the point O is the *torque about the axis*:

$$\vec{M}_Z = \left[\vec{r}, \vec{F}\right]_Z. \tag{2.81}$$

If two or more forces act on a rigid object (fig.2.13), each tends to produce rotation about the axis through *O*. The *net* torque about an axis through *O* is

$$\vec{M} = \sum_i \vec{M}_i. \tag{2.82}$$

For the case on the figure, the net torque is

$$\vec{M} = \vec{M}_1 + \vec{M}_2 + \vec{M}_3 = \left[\vec{r}_1, \vec{F}_1\right] + \left[\vec{r}_2, \vec{F}_2\right] + \left[\vec{r}_3, \vec{F}_3\right]. \tag{2.83}$$

Vector $\vec{M}$ is a pseudovector and its direction gets out conditionally using the *right hand rule*: if we put our fingers in the direction of $\vec{r}$, and turn them to the direction of $\vec{F}$, the thumb points the direction of the torque. For calculation of the torque magnitudes, we can use the convention that the sign of the torque resulting from a force is positive if the turning tendency of the force is counterclockwise and negative if the turning tendency is clockwise. Hence, for the examined case

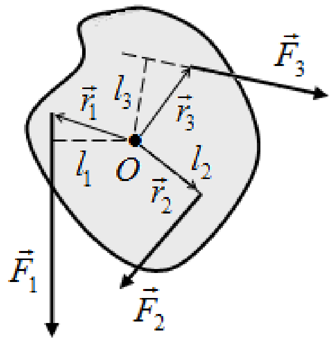


Figure 2.13

$$M = M_1 - M_2 - M_3 = F_1 l_1 - F_2 l_2 - F_3 l_3 \; . \tag{2.84}$$

A ***force couple*** (*coupled forces*) is two forces of equal magnitudes acting in opposite directions in the same plane but not same point (fig. 2.14). These two forces always have a turning effect, or moment, called a ***torque*** of the couple.

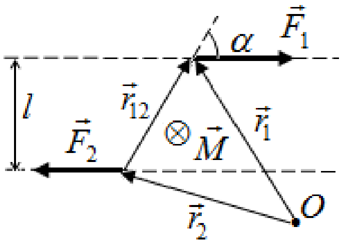


Figure 2.14

Forces $\vec{F}_1$ and $\vec{F}_2$ are separated by a perpendicular distance l called the ***arm of the couple.***

The torque of the coupled forces about a point O with regard to $\vec{F}_1 = -\vec{F}_2$ is

$$\vec{M} = \left[\vec{r}_1, \vec{F}_1\right] + \left[\vec{r}_2, \vec{F}_2\right] = \left[\vec{r}_1, \vec{F}_1\right] - \left[\vec{r}_2, \vec{F}_1\right] = \left[\left(\vec{r}_1 - \vec{r}_2\right), \vec{F}_1\right] = \left[\vec{r}_{12}, F_1\right], \tag{2.85}$$

$$\vec{M} = \left[\vec{r}_{12}, F_1\right]. \tag{2.86}$$

The magnitude of the torque of the couple is

$$M = r_{12} \cdot F \cdot \sin\alpha = F \cdot l \, . \tag{2.87}$$

Since the moment of a couple depends only on the distance between the forces, the moment of a couple is a free vector. It can be moved anywhere on the body and have the same external effect on the body.

The torque of the coupled forces can be added using the same rules as adding any vectors.

Note, that for any system of particles the sum of the moments of all internal forces is equal to zero since they are force couples with "zero" arms.

$$\sum_i \left(\vec{M}_{\text{inernal}}\right)_i = 0. \qquad (2.88)$$

2.5.2. Equilibrium of the objects

An object in mechanical equilibrium is stable, without changes in its motion. ***Mechanical equilibrium*** is a state wherein no physical changes occur; it is state of steadiness.

The ***translational equilibrium*****:** sum of all forces is equal to zero, that is, the net force

$$\vec{F} = \sum_i \vec{F}_i = 0. \qquad (2.89)$$

An object may be rotating, even rotating at a changing rate, but may be in translational equilibrium if the acceleration of the center of mass of the object is still zero.

$$\vec{M} = \sum_i \vec{M}_i = 0. \qquad (2.90)$$

The ***rotational equilibrium:*** the sum of the torques is equal to zero**.** In other words, there is no net torque on the object.

Objects at rest are said to be in ***static equilibrium***. Objects moving at constant speed in a straight-line path are said to be in ***dynamic equilibrium***. Both the static and dynamic equilibriums are attained by satisfying both of the conditions (2.89) and (2.90) at the same time.

Equilibrium may be *stable*, *unstable* or *neutral* according to whether the potential energy is minimum, maximum or constant: 1. tend to bring back to its original position if potential energy is a minimum, corresponding to stable equilibrium (fig. 2.15, *a*); 2. tend to move it farther away if potential energy is maximum, corresponding unstable equilibrium (fig. 2.15, *b*), and 3. tend to move either way if potential energy is a constant corresponding to neutral equilibrium (fig. 2.15, *c*).

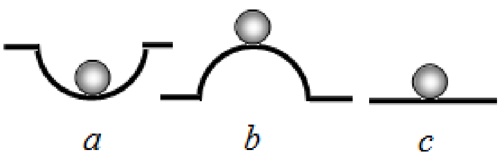


Figure 2.15

2.5.3. Angular momentum

An ***angular momentum*** of a particle with respect to origin O (fig. 2.16) is a vector quantity defined as

$$\vec{L}=[\vec{r},\vec{p}], \tag{2.91}$$

where $\vec{r}$ is a position vector of the particle having linear momentum $\vec{p}=m\vec{v}$.

Its magnitude is

$$L=r\cdot p\cdot\sin\alpha=r\cdot m\cdot v\cdot\sin\alpha=mvl, \tag{2.92}$$

where α is the smaller angle between $\vec{r}$ and $\vec{p}$ when these two vectors are tail to tail. The projection of the vector $\vec{L}$ on any axis z containing the point O is the *angular momentum about the axis* z:

$$\vec{L}=[\vec{r},\vec{p}]_Z. \tag{2.93}$$

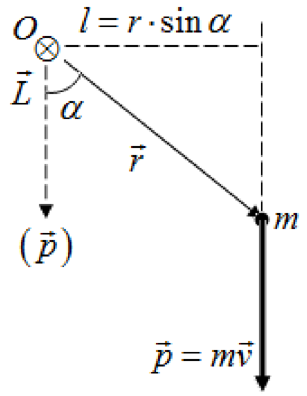


Figure 2.16

To find the direction of the angular momentum $\vec{L}$ we slide the vector $\vec{p}$ until its tail is at the origin O. Then we use the right-hand rule for vector products, sweeping the fingers from $\vec{r}$ into $\vec{p}$. The outstretched thumb then shows the direction $\vec{L}$.

The positive direction may be consistent with the counterclockwise rotation of position vector $\vec{r}$ about the z-axis, as the particle moves. A negative direction of $\vec{L}$ would be consistent with a clockwise rotation of about the z axis.

$$[L]=\mathrm{kg}\cdot\mathrm{m}^2\cdot\mathrm{s}^{-1}.$$

It is important to note that the angular momentum has meaning only with respect to a specified origin and its direction is always perpendicular to the plane formed by the position and linear momentum vectors $\vec{r}$ and $\vec{p}$.

The change in the angular momentum of the particle can be obtained by differentiating (2.91) with respect to time t:

$$\frac{d\vec{L}}{dt}=\frac{d}{dt}\left[\vec{r},m\vec{v}\right]=\left[\frac{d\vec{r}}{dt},m\vec{v}\right]+\left[\vec{r},m\frac{d\vec{v}}{dt}\right]=\left[\frac{d\vec{r}}{dt},m\vec{v}\right]+\left[\vec{r},\vec{F}\right]=$$
$$=\underbrace{\cancel{\left[\vec{v},m\vec{v}\right]}}_{0}+\left[\vec{r},\vec{F}\right]=\vec{M}, \tag{2.94}$$

$$\frac{d\vec{L}}{dt}=\vec{M}\,. \tag{2.95}$$

The rate of change of the angular momentum depends on the net torque of the forces that act on the object.

2.5.4. The law of conservation of angular momentum

Let's consider a system consisting of k particles, where both internal and external forces are acting. An angular momentum of this system with respect to the point O is the vector sum of the angular momenta of all particles of a system:

$$\vec{L}=\sum_k \vec{L}_k=\sum_k\left[\vec{r}_k,\vec{p}_k\right]. \tag{2.96}$$

With time, the angular momentum of each particle may change because of interactions between the particles or with the outside. The resulting change in the angular momentum of the system may be found by taking the time derivative of (2.96). Thus,

$$\frac{d\vec{L}}{dt}=\sum_k\frac{d\vec{L}_k}{dt}. \tag{2.97}$$

From (2.95), we see that $\frac{d\vec{L}_k}{dt}$ is equal to the net torque $\vec{M}_{\text{net}}$ on the k-th particle.

$$\frac{d\vec{L}}{dt}=\sum_k \vec{M}_k=\vec{M}_{\text{net}}\,. \tag{2.98}$$

That is, the rate of change of the system's angular momentum is equal to the vector sum of the torques on its individual particles. These torques include *internal torques* (due to forces between the particles) and *external torques* (due to forces on the particles from bodies external to the system). However, the forces between the particles always come in third-law force pairs so their torques sum to zero $\vec{M}_{\text{int}} = 0$. Thus, the only torques, that can change the total angular momentum of the system, are the external torques $\vec{\mathrm{M}}_{\text{ext}}$ acting on the system.

$$\frac{d\vec{L}}{dt} = \vec{M}_{\text{net}} = \cancel{\vec{\mathrm{M}}_{\text{int}}} + \vec{\mathrm{M}}_{\text{ext}}. \qquad (2.99)$$

If $\vec{\mathrm{M}}_{\text{ext}}$ is net external torque (the vector sum of all external torques on all particles in the system) then

$$\frac{d\vec{L}}{dt} = \vec{M}_{\text{ext}}. \qquad (2.100)$$

is the ***Newton's Second Law*** in angular form. It states that the net external torque acting on a system of particles is equal to the time rate of change of the system's total angular momentum

If no external forces act on a system of particles or if the net external torque is equal to zero $\vec{\mathrm{M}}_{\text{ext}} = 0$, the total angular momentum of the system is conserved.

$$\vec{L} = \text{const}. \qquad (2.101)$$

The net angular momentum of closed system remains constant, no matter what changes take place within the system. It is the ***law of conservation of angular momentum***.

2.6. MOMENT OF INERTIA. NEWTON'S 2ND LAW FOR ROTATION

2.6.1. Newton's 2nd law for rotation. Moment of inertia of a point mass

Let the point *m* move with acceleration $\vec{a}$ along a curvilinear path under the action of force $\vec{F}$. Using Newton's Second Law (2.3) and relationship between linear and angular accelerations (1.35) $\vec{a} = [\vec{\varepsilon}, \vec{r}]$, we can write that torque is

$$\begin{aligned} \vec{M} &= [\vec{r}, \vec{F}] = [\vec{r}, m\vec{a}] = [\vec{r}, m[\varepsilon, \vec{r}]] = m[\vec{r}, [\varepsilon, \vec{r}]] = \\ &= m\left\{\vec{\varepsilon}(\vec{r}, \vec{r}) - \vec{r}\,\underbrace{\cancel{(\vec{r}, \vec{\varepsilon})}}_{0}\right\} = mr^2\vec{\varepsilon} = I\vec{\varepsilon}, \end{aligned} \qquad (2.102)$$

$$\vec{M} = I\vec{\varepsilon}. \qquad (2.103)$$

This expression is said to be ***Newton's 2nd law for rotation***.

The quantity

$$I = mr^2. \tag{2.104}$$

tells us how the mass of the rotating particle is located relatively the axis of rotation. This quantity is called the ***moment of inertia*** (or ***rotational inertia***) of a *point mass*. It is the rotational equivalence of mass. The moment of inertia plays the same role for rotation as the mass does for a translational motion; it describes the resistance of a body to a change of its state of motion, hence, it is the measure of inertia

$[I]$ = kg·m².

2.6.2. Moment of inertia of a rigid body

Let's examine a rigid object of arbitrary shape rotating about a fixed axis passing through a point O. The object can be regarded as a collection of particle of mass Δm_i. The moment of inertia of the object depends on the masses of the particle making up the object and their distances from the rotation axis.

$$I = \sum_i \Delta m_i \cdot r_i^2. \tag{2.105}$$

If a rigid body consists of a great many adjacent particles (it is continuous) we replace the sum with an integral (fig. 2.17). Thus, for rigid body with continuously distributed mass the ***moment of inertia*** is defines as

$$I = \int r^2 dm. \tag{2.106}$$

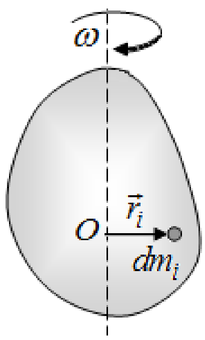


Figure 2.17

The r_i and r in expressions (2.105) and (2.106) represent the perpendicular distance from axis of rotation to each mass element in the body. The integration in (2.106) is carried out over the entire body so as to include every mass element.

The moment of inertia is an *additive* quantity. In other words, if the object can be split into components, the moment of inertia of the object as a whole about some axis is the sum of the moments of inertia of each of the components about that same axis.

In the case of the body of arbitrary shape, the moment of inertia is very hard to calculate. But the evaluation of the integral is easy in cases where mass of the body is evenly distributed about axis. This axis of symmetry passes through the center of mass of the regular body and is called "principal axis", a term which includes all axes of symmetry of objects. A *principal axis* may be simply defined as one about which no net torque is needed to maintain rotation at a constant angular velocity.

2.6.3. The calculation of moments of inertia

1. *Uniform solid cylinder* about its longitudinal axis of symmetry passing through its center of mass.

It is convenient to divide the cylinder (fig. 2.18) into many cylindrical shells each having radius r, thickness dr, and length h. Distribution of mass throughout the rigid body is characterized by volume density which is generally determined as

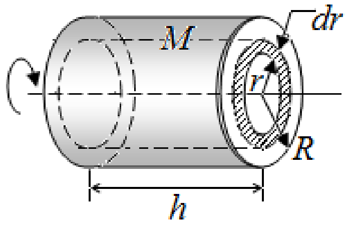


Figure 2.18

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} = \frac{dm}{dV}. \quad (2.107)$$

If the density of cylinder is ρ, the volume dV of each shell is cross-sectional area dS multiplied by its length l

$$dV = h \cdot dS = h \cdot 2\pi r \cdot dr. \quad (2.108)$$

With regard of (2.107) and (2.108), the mass element dm can be expressed in terms of an infinitesimal radial thickness dr as

$$dm = \rho dV = \rho \cdot 2\pi rh \cdot dr. \quad (2.109)$$

Substitute (2.109) into (2.106) and take the integral between limits 0 and R

$$I = \int r^2 dm = \int_0^R r^2 \rho 2\pi rh \cdot dr = 2\pi\rho h \int_0^R r^3 dr = 2\pi\rho h \frac{r^4}{4}\bigg|_0^R = \frac{\pi\rho h R^4}{2} = \frac{mR^2}{2}. \quad (2.110)$$

The moment of inertia of the cylinder (disk) of mass m and radius R is

$$I = \frac{mR^2}{2}. \quad (2.111)$$

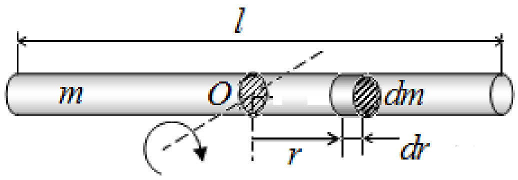


Figure 2.19

2. Moment of inertia of ***uniform rod*** with negligible thickness about its perpendicular bisector.

Let us consider a small element dr along the length which is situated at linear distance r from the axis passing through its center of mass (fig. 2.19). The linear density $\lambda = \frac{m}{l}$, where m and l are the mass and the length of the rod, is the appropriate density in this case. Elemental mass dm is expressed as

$$dm = \lambda \cdot dr = \frac{m}{l} dr . \qquad (2.112)$$

Substitution of (2.112) and integrating from $-\frac{l}{2}$ to $\frac{l}{2}$ gives

$$I = \int_{-\frac{l}{2}}^{\frac{l}{2}} r^2 \frac{m}{l} dr = \frac{m}{l} \cdot \frac{r^3}{3} \Bigg|_{-\frac{l}{2}}^{\frac{l}{2}} = \frac{m}{3l} \left(\frac{l^3}{8} - \frac{(-l)^3}{8} \right) = \frac{ml^2}{12} . \qquad (2.113)$$

The moment of inertia of the uniform rod of mass m and the length l is

$$I = \frac{ml^2}{12} . \qquad (2.114)$$

3. The moment of inertia of the ***thin hoop*** or the ***hollow (thin-walled) cylinder*** (fig. 2.20, *a*) of mass m and radius R about the perpendicular axis passing through the center is

$$I = mR^2 . \qquad (2.115)$$

4. The moment of inertia of the ***solid sphere*** (fig. 2.20, *b*) of mass m and radius R about the axis passing through the center is

$$I = \frac{2}{5} mR^2 . \qquad (2.116)$$

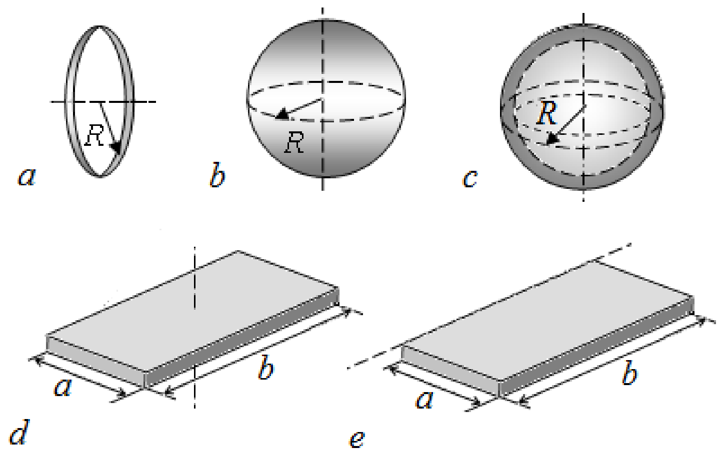


Figure 2.20

5. The moment of inertia of the ***thin-walled hallow sphere*** (fig. 2.20, *c*) of mass m and radius R about the axis passing through the center is

$$I = \frac{2}{3} mR^2. \tag{2.117}$$

6. The moment of inertia of the ***rectangular plate*** (fig. 2.20, *d*) of mass m and sides a and b about the axis passing through its center is

$$I = \frac{m(a^2 + b^2)}{12}. \tag{2.118}$$

7. The moment of inertia of the ***rectangular plate*** (fig. 2.20, *e*) of mass m and sides a and b about the axis passing along the edge is

$$I = \frac{ma^2}{3}. \tag{2.119}$$

The ***parallel axis theorem*** (***Steiner's theorem***): If we know the moment of inertia I_0 of the body about the axis that passes through its center of mass, then the moment of inertia I_x about any new axis that is parallel to the axis through the center of mass is

$$I_x = I_0 + mx^2, \tag{2.120}$$

where m is the mass of the body, and x is the perpendicular distance between two axes.

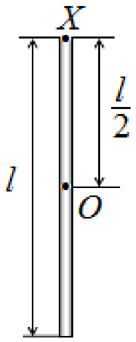


Figure 2.21

For example, the moment of inertia of the rod about the axis perpendicular to the rod through one end (fig. 2.21) is

$$I_x = I_0 + mx^2 = \frac{ml^2}{12} + m\left(\frac{l}{2}\right)^2 = \frac{ml^2}{3} .$$

The ***perpendicular axis theorem*** for planar objects states that the moment of inertia about an axis perpendicular to the plane is the sum of the moments of inertia of two perpendicular axes trough the same point in the plane of the object.

$$I_Z = I_X + I_Y . \qquad (2.121)$$

The utility of this theorem goes beyond that of calculating momenta of strictly planar objects. It is a valuable tool in building up of the momenta of inertia of three dimensional objects such as cylinders by breaking them up into planar disks and summing the momenta of inertia of the composite disks.

2.6.4. Rotational kinetic energy. Work done by torque

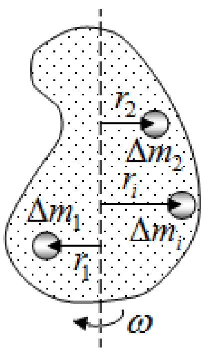


Figure 2.22

For calculation of the kinetic energy of a rotating object, let us consider it as a system of particles, and assume it rotates about a fixed axis with an angular speed ω (fig. 2.22). The particle of mass Δm_i located at a distance r_i from the rotation axis moves with tangential speed $v_i = \omega \cdot r_i$. The kinetic energy of the individual particle is

$$\left(\Delta W_k\right)_i = \frac{\Delta m_i \cdot v_i^2}{2} = \frac{\Delta m_i \cdot \omega^2 \cdot r_i^2}{2} . \qquad (2.122)$$

The *total* kinetic energy of the rotating rigid object is the sum of the kinetic energies of the individual particles

$$W_k = \sum_i \left(\Delta W_k\right)_i = \frac{\Delta m_i \cdot \omega^2 \cdot r_i^2}{2} = \frac{\omega^2}{2}\sum_i \Delta m_i \cdot r_i^2 = \frac{I\omega^2}{2} . \qquad (2.123)$$

The ***rotational kinetic energy*** is

$$W_k = \frac{I\omega^2}{2}. \tag{2.124}$$

The total ***kinetic energy of rolling object*** is the sum of the translational kinetic energy *of* the center of mass and the rotational kinetic energy *about* the center of mass. For example, the wheel of moving automobile turns around its axis, and the axis moves along parallel to the road.

$$W_k = W\frac{mv^2}{2} + \frac{I\omega^2}{2}. \tag{2.125}$$

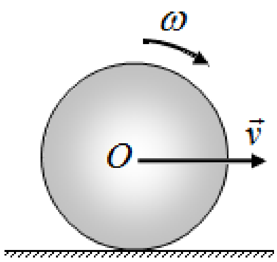


Figure 2.23

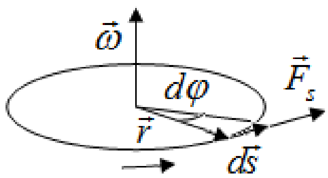


Figure 2.24

If the external force $\vec{F}$ is applied to the body at the point at distance r from the axis of rotation and turns this body through an angle $d\varphi$, the ***work done by the torque*** is

$$dA = F_S ds = F_S r \cdot d\varphi = M_Z d\varphi = M_\omega d\varphi,$$

$$dA = M_\omega d\varphi. \tag{2.126}$$

2.6.5. Corresponding variables and relations for translational and rotational motion

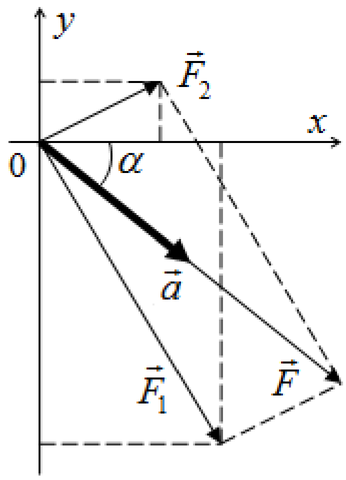
Table 1 – Translation/rotational parallels

Transitional motion	***Rotational motions***
Distance s	Angular position φ
Linear velocity $v = \frac{ds}{dt}$	Angular velocity $\omega = \frac{d\varphi}{dt}$
Linear acceleration $a = \frac{dv}{dt}$	Angular acceleration $\varepsilon = \frac{d\omega}{dt}$
Mass m	Moment of inertia $I = mr^2$
Linear momentum $p = m\vec{v}$	Angular momentum $\vec{L} = I\vec{\omega}$
Force $\vec{F}$	Torque $\vec{M}$
Newton's 2nd law $\vec{F} = \frac{d\vec{p}}{dt} = m\vec{a}$	Newton's 2nd law for rotation $\vec{M} = \frac{d\vec{L}}{dt} = I\vec{\varepsilon}$
Kinetic energy $W = \frac{mv^2}{2}$	Kinetic energy $W = \frac{I\omega^2}{2}$
Work $dA = F_S ds = F_v ds$	Work $dA = M_{\omega} d\varphi$
Power $P = Fv$	Power $P = M\omega$

PROBLEMS

Problem 2.1

Two forces $\vec{F}_1 = (3\vec{i} - 5\vec{j})$ *and* $\vec{F}_2 = (2\vec{i} + \vec{j})$ *N are applied to the particle of a mass* $m = 1.5$ *kg. Find the acceleration* $\vec{a}$ *of this particle.*



Solution

Newton's 2 Law gives

$$m\vec{a} = \vec{F},$$

where $\vec{F}$ is a net force (the vector sum of all forces that act on the particle).

$$\vec{a} = \frac{\vec{F}}{m} = \frac{(3\vec{i} - 5\vec{j}) + (2\vec{i} + \vec{j})}{1.5} = \left(\frac{3+2}{1.5}\right)\vec{i} + \left(\frac{-5+1}{1.5}\right)\vec{j} = (3.33\vec{i} - 2.67\vec{j}) \text{ m/s}^2.$$

Since the projections on the x and y axes are equal to 3.33 m/s² and 2.67 m/s², respectively, the magnitude of the acceleration can be determined as

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{3.33^2 + 2.67^2} = 4.27 \text{ m/s}^2.$$

The vector of acceleration is directed at the angle

$$\alpha = \arctan\left(\frac{2.67}{3.33}\right) = \arctan 0.8 = 38.7° \text{ below the positive } x\text{-axis.}$$

Problem 2.2

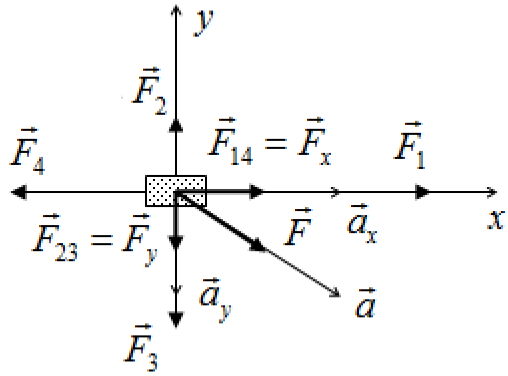
Forces of $F_1 = 85$ *N to the east,* $F_2 = 25$ *N to the north,* $F_3 = 45$ *N to the south,* $F_4 = 55$ *N to the west are simultaneously applied to a box of mass* 14 *kg. Find the magnitude of the box's acceleration?*

Solution

Two forces $\vec{F}_1$ and $\vec{F}_4$ act in the x-direction. Their vector sum is $\vec{F}_x = \vec{F}_{14} = \vec{F}_1 + \vec{F}_4$, and the magnitude is

$$F_x = F_{14} = F_1 - F_4 = 85 - 45 = 40 \text{ N}.$$

Two other forces are directed along y-axis.



$$\vec{F}_y = \vec{F}_{23} = \vec{F}_2 + \vec{F}_3,$$

$$F_y = F_{23} = F_3 - F_2 = 45 - 25 = 20 \text{ N}.$$

The net applied to the box is $\vec{F} = \vec{F}_{14} + \vec{F}_{13} = \vec{F}_x + \vec{F}_y$. Its magnitude using Pythagorean Theorem is

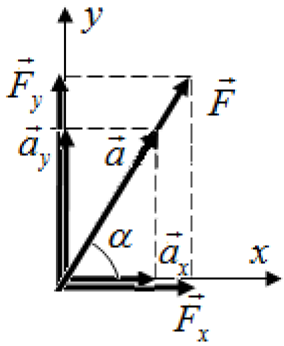
$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{40^2 + 20^2} = 44.7 \text{ N}.$$

According to Newton's 2nd Law, $\vec{F} = m\vec{a}$, therefore,

$$a = \frac{F}{m} = \frac{44.7}{14} = 3.2 \text{ m/s}^2.$$

Problem 2.3

A 5-kg object undergoes an acceleration given by $\vec{a} = (3\vec{i} + 5\vec{j})$ *m/s². Find the magnitude and direction of the resultant force acting on it.*



Solution

The resultant force from the Newton's 2nd Law is

$$\vec{F} = m\vec{a} = 5 \cdot (3\vec{i} + 5\vec{j}) = (15\vec{i} + 25\vec{j}) \text{ N}.$$

The components of the force are $F_x = 15$ N and $F_y = 25$ N.

The magnitude of the net force is

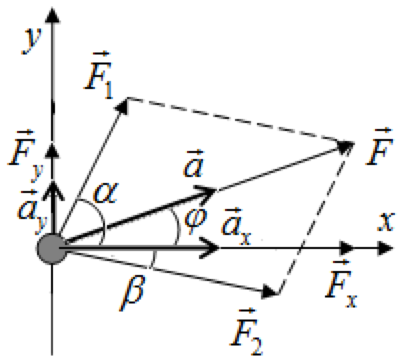
$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{15^2 + 25^2} = 29.1 \text{ N}$$

Direction of the force is determined by the angle α that the acceleration and the force make with the positive direction of the x-axis

$$\alpha = \arctan(F_x/F_y) = \arctan(15/25) = 30.9°.$$

Problem 2.4

A hockey puck with the mass of 0.15 *kg slides on the horizontal, frictionless surface of an ice rink. Two hockey-players strike the puck simultaneously by their sticks, exerting the forces on the puck shown in Figure. The force* F_1 *has a magnitude of* 6 *N, and it makes the angle* $60°$ *above the* $+x$*-axis. The force* F_2 *has a magnitude of* 9 *N and direction* $20°$ *below the* $+x$*-axis. Determine both the magnitude and the direction of the puck's acceleration assuming that the puck slides on the horizontal, frictionless surface of the ice rink.*



Solution

Firstly, resolve the force vectors into components. The net force acting on the puck in the x-direction is

$$\sum F_x = F_{1x} + F_{2x} = F_1 \cdot \cos\alpha + F_2 \cdot \cos\beta = 6 \cdot \cos 60^0 + 9 \cdot \cos(-20^0) = 11.5 \text{ N}.$$

The y-component of the net force is

$$\sum F_y = F_{1y} + F_{2y} = F_1 \cdot \sin\alpha + F_2 \cdot \sin\beta = 6 \cdot \sin 60^0 + 9 \cdot \sin(-20^0) = 2.1 \text{ N}.$$

Using the Newton's 2nd Law in component form, x- and y-components of the puck acceleration may be found.

$$a_x = \frac{\sum F_x}{m} = \frac{11.5}{0.15} = 76.7 \text{ m/s}^2,$$

$$a_y = \frac{\sum F_y}{m} = \frac{2.1}{0.15} = 14 \text{ m/s}^2.$$

The magnitude of the puck's acceleration is

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{76.7^2 + 14^2} = 77.9 \text{ m/s}^2,$$

and its direction makes an angle φ with the positive direction of x-axis

$$\varphi = \arctan\left(a_y / a_x\right) = \arctan\left(14/76.7\right) = 7°.$$

Problem 2.5

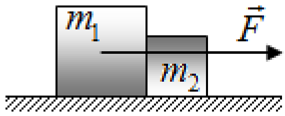
Two blocks of masses m_1 =5 *kg and* m_2 =3 *kg are place in contact with each other on a frictionless, horizontal surface. A constant force* F =20 *N is applied to the block* m_1 *in horizontal direction. a) Find the magnitude of the acceleration of the system, and b) the magnitude of the contact force between the two blocks.*

Solution

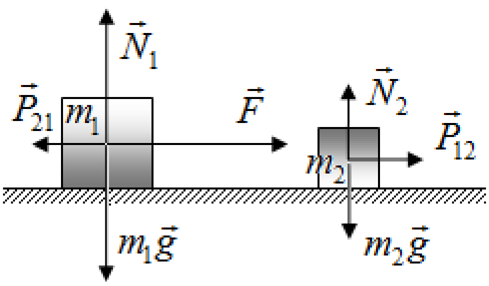
a) Here we have a force applied to a system consisting of two masses m_1 and m_2.

$$F = M \cdot a = (m_1 + m_2) \cdot a,$$

$$a = \frac{F}{m_1 + m_2} = \frac{20}{5+3} = 2.5 \text{ m/s}^2.$$



b) The contact force is internal to the system of two blocks. Thus, we cannot find this force by modeling the whole system (the two blocks) as a single particle. We must now treat each of the two blocks individually by categorizing each as a particle subject to a net force. The only horizontal force acting on m_2 is the contact force $\vec{P}_{12}$ (the force exerted by m_1 on m_2), which is directed to the right. Applying Newton's 2nd Law to m_2 gives



$$P_{12} = m_2 \cdot a.$$

Substitution of the value of the acceleration obtained in the previous part into the expression for P_{12} gives

$$P_{12} = m_2 a = m_2 \left(\frac{F}{m_1 + m_2} \right) = 3 \left(\frac{20}{5+3} \right) = 7.5 \text{ N}.$$

This result indicates that the contact force P_{12} is less than the applied force F. This is consistent with the fact that the force required to accelerate the second block alone must be less than the force required to produce the same acceleration for the two-block system.

The horizontal forces acting on m_1 are the applied force $\vec{F}$ to the right and the contact force P_{21} to the left (the force exerted by m_2 on m_1).

Applying Newton's 2nd Law to m_1 gives

$$m_1 a = F - P_{21}.$$

Substituting the expression for acceleration, we obtain

$$P_{21} = F - m_1 a = F - m_1\left(\frac{F}{m_1 + m_2}\right) = 20 - 5\left(\frac{20}{5+3}\right) = 7.5\,\text{N}.$$

Thus, magnitudes of contact forces are equaled. This result is quite obvious from the Newton's 3rd Law: since P_{21} is the reaction to P_{12}, they are equal in value, $P_{21} = P_{12}$.

Problem 2.6

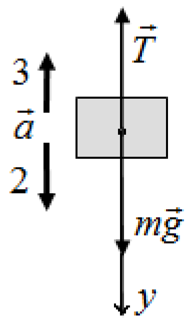
The load of mass 1 kg is suspended by the thread. Find the tension $\vec{T}$ if (a) the thread is at rest; (b) the thread is moving downwards at acceleration $a = 5$ m/s^2; (c) the thread is moving upwards at acceleration $a = 5$ m/s^2.

Solution

This problem deals with forces (gravity, tension) and accelerations. That suggests we should apply Newton's 2nd Law. To apply this law we draw the forces and accelerations for all cases that are under consideration. Since the load has mass, there is gravity $m\vec{g}$. The load is suspended by the thread, so there is the tension $\vec{T}$.

$$m\vec{a} = m\vec{g} + \vec{T}.$$

Let y-axis is directed downwards. Then the projections of this equation for different cases are following.



(a) When the object is at rest the acceleration $\vec{a} = 0$, so

$$0 = mg - T,$$

$$T = mg = 1 \cdot 9.8 = 9.8\,\text{N}.$$

(b) Acceleration $\vec{a}$ is directed downwards.

$$ma = mg - T,$$

$$T = m(g - a) = 1(9.8 - 5) = 4.8\text{N};$$

(c) Acceleration $\vec{a}$ is directed upwards.

$$-ma = mg - T,$$

$$T = m(g + a) = 1(9.8 + 5) = 14.8 \text{ N}.$$

Problem 2.7

A tension of 6000 *N is experienced by the elevator cable of an elevator moving upwards with an acceleration of* 2 *m/s*^{2}. *What is the mass of the elevator?*

Solution

The equation of the elevator motion according to the Newton's 2nd Law is

$$m\vec{a} = m\vec{g} + \vec{T}.$$

Using the figure to previous problem (the third case), the projections of the forces and acceleration on the axis are

$$-ma = mg - T.$$

The mass of elevator is

$$m = \frac{T}{g + a} = \frac{6000}{9.8 + 2} = 508 \text{ kg}.$$

Problem 2.8

A box is pulled with 20 *N force making angle* $\alpha = 60°$ *with horizontal. Mass of the box is* 2 *kg. Find the acceleration of the box if a) the surface is frictionless, and b) coefficient of friction is* $\mu = 0.1$.

Solution

The forces acting on the box are shown in the figures below.

a) The body is moving under the action of following forces: gravitational force $m\vec{g}$, normal force $\vec{N}$, and force $\vec{F}$.

The equation of motion according the Newton's 2nd Law is

$$m\vec{a} = m\vec{g} + \vec{N} + \vec{F} .$$

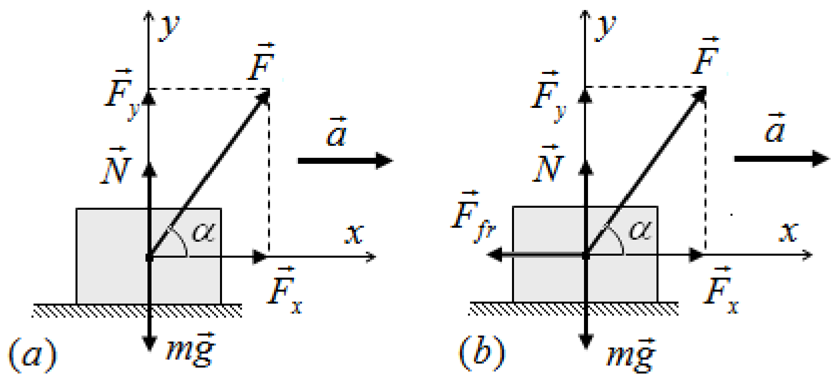
Let's find the projections of these forces on x- and y-axes:

$$\begin{cases} ma = F_x, \\ 0 = -mg + N + F_y, \end{cases}$$

$$ma = F_x = F\cos\alpha .$$

Finally,

$$a = \frac{F\cos\alpha}{m} = \frac{20\cdot 0.5}{2} = 5 \text{ m/s}^2.$$



b) When friction has to be taken into account, the equation of motion according the Newton's 2nd Law is

$$m\vec{a} = m\vec{g} + \vec{N} + \vec{F} + \vec{F}_{fr} .$$

Projections of this equation on x- and y-axes are

$$\begin{cases} ma = F_x - F_{fr}, \\ 0 = -mg + N + F_y. \end{cases}$$

Since $F_{fr} = \mu N$, and $N = mg - F_y = mg - F\cdot\sin\alpha$, the acceleration is

$$a = \frac{F_x - F_{fr}}{m} = \frac{F\cos\alpha - \mu N}{m} = \frac{F\cos\alpha - \mu\left(mg - F\cdot\sin\alpha\right)}{m} =$$
$$= \frac{F\left(\cos\alpha + \mu\sin\alpha\right)}{m} - \mu g.$$

Substituting the numerical values, we'll obtain the acceleration

$$a = \frac{20 \cdot (\cos 60° + 0.1 \cdot \sin 60°)}{2} - 0.1 \cdot 9.8 = 4.87 \text{ m/s}^2.$$

Problem 2.9

The body under the effect of applied force F = 10 N is moving according the dependence $s = A - Bt + Ct^2$, where C = 1 m/s^2. Find the mass of the body.

Solution

Let find the acceleration of the body by differentiation of $s = A - Bt + Ct^3$. The first derivative gives the time dependence of speed

$$v(t) = \frac{ds}{dt} = -B + 2Ct.$$

Taking derivative of speed, we obtain the acceleration

$$a = \frac{dv}{dt} = 2C = 2 \cdot 1 = 2 \text{ m/s}^2.$$

Using the Newton's 2nd Law $\vec{F} = m\vec{a}$, we can find the mass of the body

$$m = \frac{F}{a} = \frac{10}{2} = 5 \text{ kg}.$$

Problem 2.10

A 0.01 kg object is moving in a plane. The x and y coordinates of the object are given by $x(t) = 2t^3 - t^2$ and $y(t) = 4t^3 + 2t$. Find the linear momentum and the net force acting on the object at t = 2 s.

Solution

The velocity of the object may be determined by differentiating of $x(t)$ and $y(t)$ dependencies.

$$v_x(t) = \frac{dx}{dt} = 6t^2 - 2t \Big|_{t=2} = 6 \cdot 2^2 - 2 \cdot 2 = 20 \text{ m/s},$$

$$v_y(t)=\frac{dy}{dt}=12t^2+2\Big|_{t=2}=12\cdot 2^2+2=50\ \text{m/s}.$$

The magnitude of velocity is $v=\sqrt{v_x^2+v_y^2}=\sqrt{20^2+50^2}=53.9$ m/s, and the linear momentum is $p=mv=0.01\cdot 53.9=0.539$ kg·m/s.

To find the net force we need to determine the acceleration of the object. The acceleration is the second derivative of coordinate of the object. Then the x and y components of acceleration are given by the following expressions

$$a_x(t)=\frac{d^2x(t)}{dt^2}=12t-2\Big|_{t=2}=12\cdot 2-2=22\ \text{m/s}^2,$$

$$a_y(t)=\frac{d^2y(t)}{dt^2}=24t\Big|_{t=2}=24\cdot 2=48\ \text{m/s}^2.$$

Then the magnitude of acceleration is

$$a=\sqrt{a_x^2+a_y^2}=\sqrt{22^2+48^2}=53\ \text{m/s}^2.$$

The net force is

$$F=ma=0.01\cdot 53=0.53\ \text{N}.$$

Problem 2.11

A body with a mass of 1 *kg is accelerated by a force F =* 2 *N. What is velocity of this body after* 5 *s of motion*?

Solution

We get expression for acceleration from Newton's 2nd Law of motion as

$$a=\frac{F}{m}.$$

The speed for accelerated motion follows the formula $v=v_0+at$, where initial speed $v_0=0$, therefore,

$$v=at=\frac{F}{m}t.$$

Substituting numbers given in the problem we get

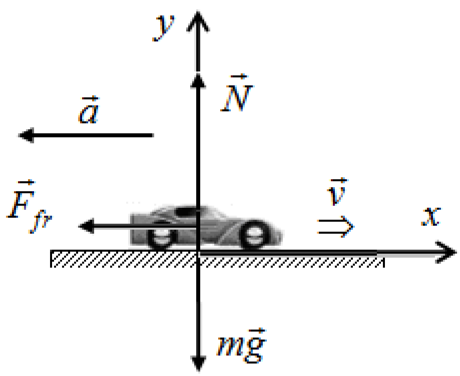
$$v = \frac{2 \cdot 5}{1} = 10 \text{ m/s.}$$

Problem 2.12

The coefficient of friction between the tires of a car and a horizontal road is $\mu = 0.55$. *a) Find the magnitude of the maximum acceleration of the car when it is braked; b) What is the shortest distance in which the car can stop if it is initially traveling at* 17 *m/s? Neglect air resistance and rolling friction.*

Solution

a) The forces acting on the car during braking are: the gravity, the normal force and the friction force. If the velocity of the car is to the right the acceleration is directed to the left. Applying Newton's 2nd Law gives



$$m\vec{a} = m\vec{g} + \vec{N} + \vec{F}_{fr}.$$

Its x- and y-projections are

$$\begin{cases} ma = F_{fr}, \\ 0 = N - mg. \end{cases}$$

Friction force is

$$F_{fr} = \mu \cdot N = \mu \cdot mg,$$

therefore,

$$ma = F_{fr} = \mu \cdot mg,$$

Solving for a and substituting numerical values, we obtain

$$a = \mu \cdot g = 0.55 \cdot 9.8 = 5.39 \text{ m/s}^2.$$

b) Using a constant-acceleration equation, relate the stopping distance s of the car to its initial speed v_0 and final speed $v = 0$, and its acceleration a:

$$s = \frac{v_0^2 - v^2}{2a} = \frac{v_0^2}{2a}.$$

The stopping distance is

$$s=\frac{17^2}{2\cdot 5.39}=26.8 \text{ m}.$$

Problem 2.13

Two blocks $m_1=15$ *kg and* $m_2=20$ *kg connected by a rope of negligible mass are being dragged by a horizontal force* F *=70 N. The coefficient of kinetic friction between each block and the surfaces is* $\mu=0.1$. *Determine the acceleration* a *of the system and the tension* T *in the rope.*

Solution

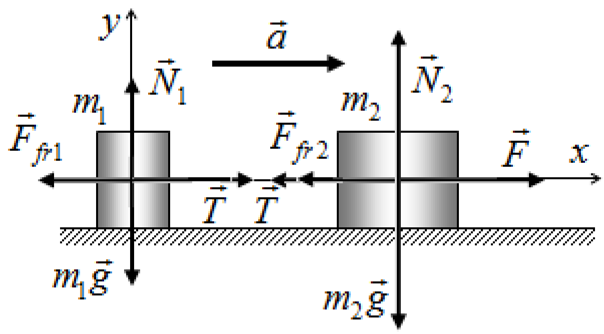
Because the string does not stretch, the blocks will have the same acceleration, and their motions may be described by equations

$$\begin{cases} m_1\vec{a}=m_1\vec{g}+\vec{F}_{fr1}+\vec{N}_1+\vec{T}, \\ m_2\vec{a}=m_2\vec{g}+\vec{F}_{fr2}+\vec{N}_2+\vec{T}+\vec{F}. \end{cases}$$

Projections of these equations on the x- and y-axes, relatively, are given by

$$x:\begin{cases} m_1a=-F_{fr1}+T, \\ m_2a=-F_{fr2}-\vec{T}+F. \end{cases}$$

$$y:\begin{cases} 0=-m_1g+N_1, \\ 0=-m_2g+N_2. \end{cases}$$



Adding x-projections from both systems, we obtain

$$a(m_1+m_2)=F-\left(F_{fr1}+F_{fr2}\right).$$

The y-projections gives $N_1=m_1g$ and $N_2=m_2g$. Taking into account that

$$F_{fr1}=\mu\cdot N_1=\mu\cdot m_1g,$$

$$F_{fr2}=\mu\cdot N_2=\mu\cdot m_2g,$$

we determine the acceleration of the system

$$a = \frac{F - \mu g\left(m_1 + m_2\right)}{m_1 + m_2} = \frac{F}{m_1 + m_2} - \mu g = \frac{70}{35} - 0.1 \cdot 9.8 = 1.02 \text{ m/s}^2.$$

The tension calculated from x-projection of the first load equation of motion is equal to

$$T = m_1 a + F_{fr1} = m_1 a + \mu \cdot m_1 g = m_1\left(a + \mu g\right) = 15\left(1.02 + 0.1 \cdot 9.8\right) = 30 \text{ N}.$$

Problem 2.14

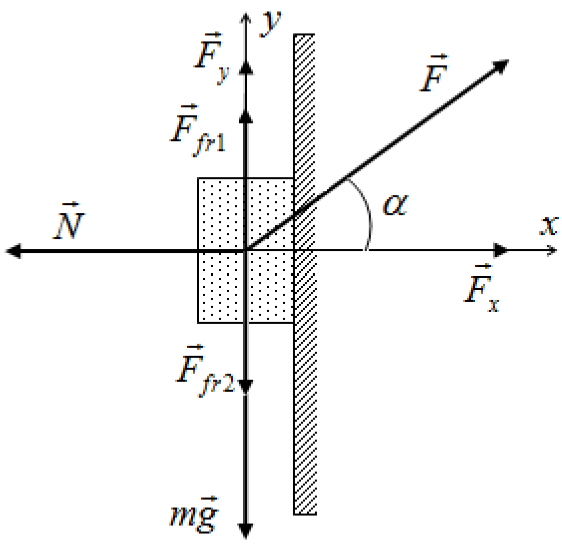
A block of mass 5 kg is pushed up against a wall by a force $\vec{F}$ that makes angle $\alpha = 40°$ with the horizontal as shown in Figure. The coefficient of static friction between the block and the wall is $\mu = 0.3$. Determine the possible values for the magnitude of F that allow the block to remain stationary.

Solution

According to Newton's 1st Law if the load is at rest the vector sum of the forces applied to it is zero:

$$m\vec{g} + \vec{N} + \vec{F}_{fr} + \vec{F} = 0.$$

Depending on the relationship between the magnitudes of forces $m\vec{g}$ and $\vec{F}$, friction force $\vec{F}_{fr}$ may be directed upwards or downwards preventing the downward and upward motion of the load, respectively. Taking into account that the system is at rest, the projections on the x and y axes are



$$\begin{cases} F_x - N = 0, \\ F_y - mg \pm F_{fr} = 0, \end{cases}$$

$$F_x - N = F_y - mg \pm F_{fr},$$

where $F_x = F \cdot \cos\alpha$, $F_y = F \cdot \sin\alpha$.

Since

$$F_{fr} = \mu \cdot N = \mu \cdot F_x = \mu \cdot F \cdot \cos\alpha\,,$$

$$F \cdot \sin\alpha - mg \pm \mu \cdot F \cdot \cos\alpha = 0$$

$$F\left(\sin\alpha \pm \mu \cdot \cos\alpha\right) = mg\,,$$

$$F = \frac{mg}{\sin\alpha \pm \mu \cdot \cos\alpha}$$

$$F = \frac{5 \cdot 9.8}{\sin 40° \pm 0.3 \cdot \cos 40°}.$$

$$F_1 = 57.0 \text{ N},$$

$$F_2 = 116.7 \text{ N}.$$

The possible values of the magnitude of force F that allow the block to remain stationary are $F_{\min} = 57$ N and $F_{\max} = 116.7$ N.

Problem 2.15

A box is placed on a plane with slope angle $\alpha = 4°$. *a) What is the static coefficient of friction needed for this box begins to move? b) Find an acceleration of the box if the coefficient of kinetic friction is* $\mu = 0.03$. *What time does it take to the box to cover the distance* 100 *m. What is its velocity in the terminal point of motion?*

Solution

Friction forces act between two bodies which are in contact but not moving or sliding with respect to each other. The friction in this case is the static friction, and the force of static friction is $F_{fr} = \mu_s \cdot N$, where μ_s is the coefficient of static friction.

Newton's 2nd Law for the box motion is

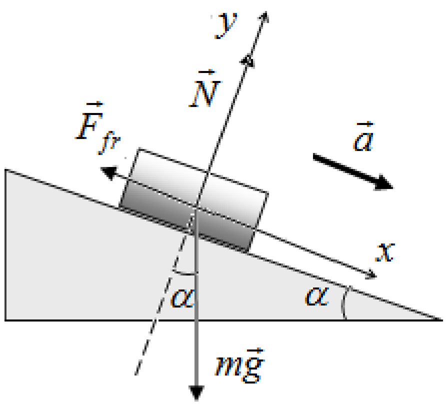
$$m\vec{a} = m\vec{g} + \vec{N} + \vec{F}_{fr}.$$

Projections of the equation on the chosen x- and y-axes are

$$\begin{cases} ma = mg\sin\alpha - F_{fr}, \\ 0 = -mg\cos\alpha + N. \end{cases}$$

a) For the first case, we have to find the coefficient of static friction μ_s. The second equation of the system gives $N = mg\cos\alpha$, then from the first equation

$$ma = mg\sin\alpha - \mu_s \cdot N = mg\sin\alpha - \mu_s \cdot mg\cos\alpha = mg\left(\sin\alpha - \mu_s \cdot \cos\alpha\right).$$



Taking into account that $a = 0$, we obtain

$$0 = g\left(\sin\alpha - \mu_s \cdot \cos\alpha\right).$$

Therefore,

$$\mu_s = \frac{\sin\alpha}{\cos\alpha} = \tan\alpha = \tan 5° = 0.08.$$

b) When two surfaces are moving with respect to one another, the friction force depends on the coefficient of kinetic friction μ_k. The coefficient of kinetic friction is typically smaller than the coefficient of static friction. According to the given data $\mu_k = 0.03 < \mu_s$, consequently, the box is sliding along the incline with acceleration

$$a = g(\sin\alpha - \mu_k \cdot \cos\alpha) = 9.8\left(0.087 - 0.03 \cdot 0.996\right) = 0.56 \text{ m/s}^2.$$

Since the object is moving at acceleration from the rest, the time of motion t and the terminal speed v may be found by means of the kinematical equations with regard for $v_0 = 0$.

$$\begin{cases} s = v_0 t + \dfrac{at^2}{2} = \dfrac{at^2}{2}, \\ v = v_0 + at = at. \end{cases}$$

$$t = \sqrt{\frac{2s}{a}} = \sqrt{\frac{2 \cdot 10}{0.56}} = 6 \text{ s},$$

$$v = at = 0.66 \cdot 6 = 3.96 \text{ m/s}.$$

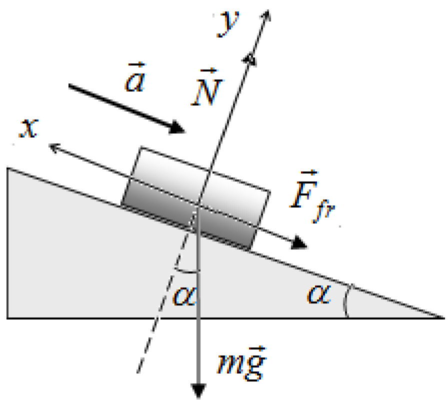
Problem 2.16

A box is sliding up an incline that makes an angle of $20°$ *with respect to the horizontal. The coefficient of kinetic friction between the box and the surface of the incline is* $\mu = 0.2$. *The initial speed of the box at the bottom of the incline is 2 m/s. How far does the box travel along the incline before coming to rest?*

Solution

The box moves upward incline with acceleration directed downwards because this motion is decelerated one. We can find the acceleration using the Newton's 2nd Law

$$m\vec{a} = m\vec{g} + \vec{N} + \vec{F}_{fr},$$



The projections on x- and y- axes are

$$\begin{cases} -ma = -mg\sin\alpha - F_{fr}, \\ 0 = -mg\cos\alpha + N. \end{cases}$$

Since

$$N = mg\cos\alpha,$$

the second equation of the system is

$$ma = mg\sin\alpha + F_{fr} = mg\sin\alpha + \mu \cdot N = mg\sin\alpha + \mu \cdot mg\cos\alpha.$$

Than the acceleration is

$$a = g(\sin\alpha + \mu\cos\alpha) = 9.8\left(0.342 + 0.2 \cdot 0.94\right) = 5.2 \text{ m/s}^2.$$

From kinematics the relation between the travelled distance s and initial v_0 and final v speeds with regard for $v = 0$ is

$$v_0 = 2as.$$

Then

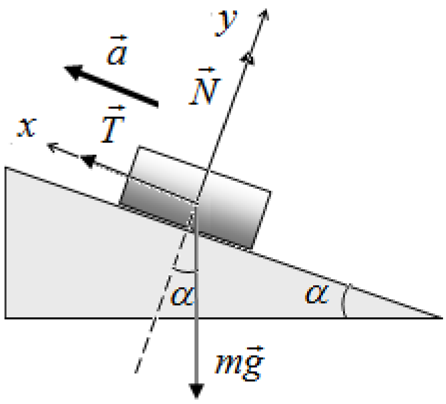
$$s = \frac{v_0^2}{2a} = \frac{4}{2 \cdot 5.2} = 0.385 \text{ m}.$$

Problem 2.17

A 10 kg block is towed up an inclined at $\alpha = 30°$ with respect to the horizontal. The rope is parallel to the incline and has a tension of 100 N. Assume that the block starts from rest at the bottom of the hill, and neglect friction. How fast is the block going after moving 40 m up the hill?

Solution

To find the speed of the block we need to find the acceleration of the block.



The motion of the block is described by the Newton's 2nd Law

$$m\vec{a} = m\vec{g} + \vec{N} + \vec{T}.$$

Projections on chosen axes are

$$\begin{cases} ma = T - mg\sin\alpha, \\ 0 = -mg\cos\alpha + N. \end{cases}$$

The acceleration of the system is given by the first equation

$$a = \frac{T - mg\sin\alpha}{m} = \frac{100 - 10\cdot 9.8\cdot \sin 30^0}{10} = 5.1 \text{ m/s}^2.$$

The kinematic equations that describe the accelerated motion of the block are

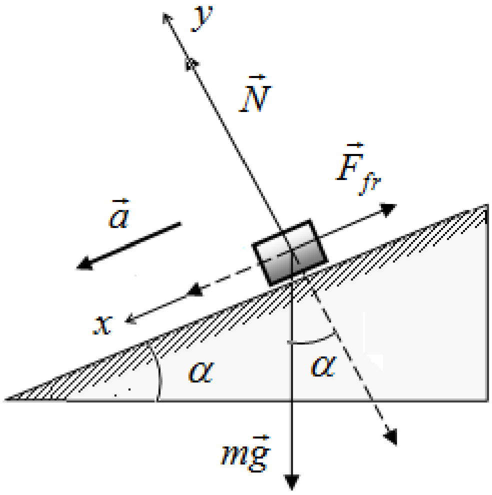
$$\begin{cases} s = v_0 t + \dfrac{at^2}{2}, \\ v = v_0 + at. \end{cases}$$

Taking into account that the initial speed $v_0 = 0$ and excluding time of motion, we obtain relationship between acceleration and travelled distance

$$v = \sqrt{2as} = \sqrt{2\cdot 5.1\cdot 40} = 20.2 \text{ m/s}.$$

Problem 2.18

A car is going at a speed of $v_0 = 30$ *km/h when it encounters a* 150 *m long slope of angle* $\alpha = 30°$. *The friction coefficient between the road and the tyre is* $\mu = 0.3$. *Show that no matter how hard the driver applies the brakes; the car will reach the bottom with a speed greater than* 100 *km/h. Take* $g = 10$ *m/s²*



Solution

The forces acting on the car during downward accelerated motion are: the gravity, normal force and friction force.

$$m\vec{a} = m\vec{g} + \vec{N} + \vec{F}_{fr}.$$

Projections on x- and y- axes are

$$\begin{cases} ma = mg \cdot \sin\alpha - F_{fr}, \\ 0 = N - mg \cdot \cos\alpha. \end{cases}$$

The brake, even the hardest one, will produce the friction force

$$F_{fr} = \mu \cdot N = \mu \cdot mg \cdot \cos\alpha.$$

Thus the acceleration is

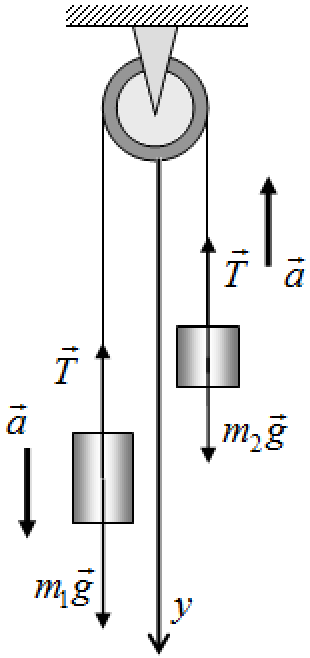
$$a = g(\sin\alpha - \mu \cdot \cos\alpha) = 10\left(\sin 30^0 - 0.3 \cdot \cos 30^0\right) = 2.4 \text{ m/s}^2.$$

The initial speed of the car is $v_0 = 30$ km/h = 8.33 m/s; the covered distance is $s = 150$ m. Using the expression from kinematics $v^2 = v_0^2 + 2as$, we obtain the final speed of the car as

$$v = \sqrt{v_0^2 + 2as} = \sqrt{8.33^2 + 2 \cdot 2.4 \cdot 150} = 28.1 \text{ m/s} = 101.2 \text{ km/h}$$

Problem 2.19

Two loads of masses 2 *kg and* 1 *kg are connected by the inextensible rope passing over a small frictionless fixed pulley. If the rope and the pulley are weightless, find the acceleration* $\vec{a}$ *of the loads and tension* $\vec{T}$ *of the rope.*



Solution

Since the rope is inextensible, the accelerations of the loads are equal in value but opposite in sign, and they are moving in opposite directions, and the tensions on both sides are the same. The equations of motion are

$$\begin{cases} m_1\vec{a} = m_1\vec{g} + \vec{T}, \\ m_2\vec{a} = m_2\vec{g} + \vec{T}. \end{cases}$$

If the axis y is directed downwards, the projections of these equations are

$$\begin{cases} m_1 a = m_1 g - T, \\ -m_2 a = m_2 g - T. \end{cases}$$

Consequently, the acceleration and tension are

$$a = \frac{g(m_1 - m_2)}{m_1 + m_2} = \frac{9.8(2-1)}{2+1} = 3.27 \text{ m/s}^2,$$

$$T = m_1(g-a) = m_1 g\left(1 - \frac{m_1 - m_2}{m_1 + m_2}\right) = \frac{2m_1 m_2 g}{m_1 + m_2} = \frac{2\cdot 2\cdot 1\cdot 9.8}{2+1} = 13.07 \text{ N}.$$

Problem 2.20

A block of mass m_1 *= 4 kg on the inclined plane of angle* $\alpha = 30°$ *(coefficient of friction* $\mu = 0.1$*) is connected by a rope over a pulley to another block of mass* m_2 *= 1 kg. What are the magnitude and direction of the acceleration of the second block?*

Solution

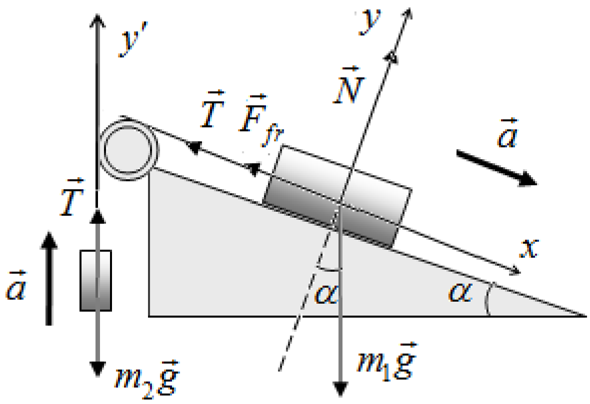
According to Newton's 2nd Law the equations of the blocks motion are

$$\begin{cases} m_1\vec{a} = m_1\vec{g} + \vec{N} + \vec{T} + \vec{F}_{fr}, \\ m_2\vec{a} = m_2\vec{g} + \vec{T}. \end{cases}$$

Let's assume that the block m_2 will move with acceleration $\vec{a}$ directed downwards. The sign of calculated acceleration will indicate where our assumptions are correct.

The projections on the *x, y*, and y' axes are following:

$$\begin{cases} m_1 a = m_1 g \sin\alpha - T - F_{fr}, \\ 0 = -m_1 g \cos\alpha + N, \\ m_2 a = -m_2 g + T. \end{cases}$$



Let's express the normal force from the second equation of system

$$N = m_1 g \cos\alpha,$$

and tension from the third equation

$$T = m_2 g + m_2 a.$$

Then substitute them into the first equation taking into account that $F_{fr} = \mu \cdot N$.

$$m_1 a = m_1 g \sin\alpha - m_2 a - m_2 g - \mu \cdot m_1 g \cos\alpha.$$

After a little of elementary algebra we get

$$a = g\frac{m_1 \sin\alpha - m_2\left(1 + \mu \cdot \cos\alpha\right)}{m_1 + m_2}.$$

Substituting numbers given in the problem we obtain

$$a = 9.8 \cdot \frac{4 \cdot 0.5 - 1\left(1 + 0.1 \cdot 0.866\right)}{4 + 1} = 1.79\ \text{m/s}^2.$$

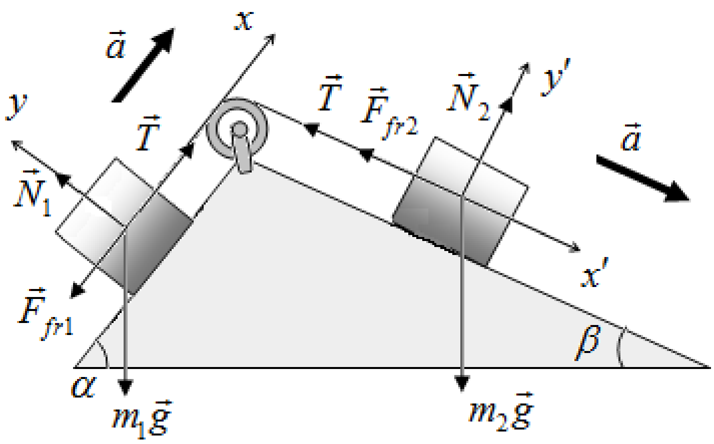
The "plus" sign tells us, that acceleration is direction downward along incline just as we have supposed. This is a general rule for a lot of problems. The negative numerical value means that the direction of the parameter found is opposite to the one which was assumed for writing equations.

Problem 2.21

Two masses $m_1 = 1$ kg and $m_2 = 10$ kg are on inclines ($\alpha = 50°$ and $\beta = 30°$) and are connected together by a string as shown in the figure. The coefficient of kinetic friction between each mass and its incline is $\mu = 0.25$. If m_1 moves up, and m_2 moves down, determine their acceleration.

Solution

We define the positive x and x' directions to be the directions of motion for each block. Equations according to the Newton's 2 Law for both objects are



$$\begin{cases} m_1\vec{a} = m_1\vec{g} + \vec{N}_1 + \vec{T} + \vec{F}_{fr1}, \\ m_2\vec{a} = m_2\vec{g} + \vec{N}_2 + \vec{T} + \vec{F}_{fr2}. \end{cases}$$

The projections on the chosen axes are

$$x:\ m_1 a = T - m_1 g \cdot \sin\alpha - F_{fr1},$$

$$x':\ m_2 a = m_2 g \cdot \sin\beta - T - F_{fr2},$$

$$y:\ 0 = -m_1 g \cdot \cos\alpha + N_1,$$

$$y':\ 0 = -m_2 g \cdot \cos\beta + N_2.$$

The friction forces with regard to the last two equations are

$$F_{fr1} = \mu \cdot N_1 = \mu \cdot m_1 g \cdot \cos\alpha,$$

$$F_{fr2} = \mu \cdot N_2 = \mu \cdot m_2 g \cdot \cos\beta.$$

Now add the x and x' projections, substitute the friction forces, and solve for acceleration

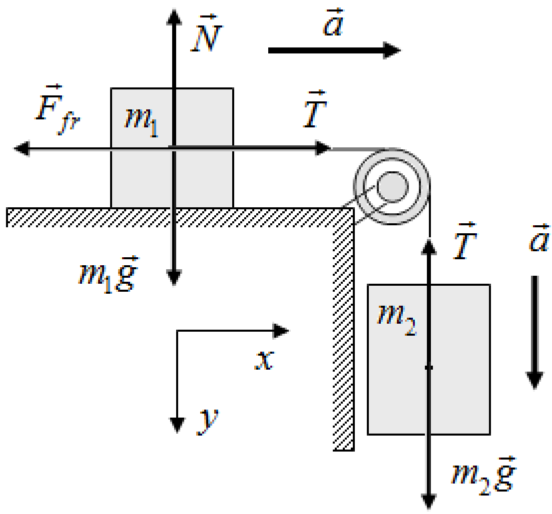
$$a = g \cdot \frac{m_2\left(\sin\beta - \mu\cdot\cos\beta\right) - m_1\left(\sin\alpha + \mu\cdot\cos\alpha\right)}{m_1 + m_2}.$$

Substitute the numerical values and obtain

$$a = 9.8 \cdot \frac{10\cdot\left(\sin 30 - 0.25\cdot\cos 30\right) - 1\cdot\left(\sin 50 + 0.25\cdot\cos 50\right)}{1+10} = 2.67\ \text{m/s}^2.$$

Problem 2.22

A 2-kg load placed on a frictionless, horizontal table is connected to a string that passes over a pulley and then is fastened to a hanging 3-kg load. a) Find the acceleration of the two loads and the tension in the string. b) Solve the problem if the coefficient of kinetic friction between the first load and the table is $\mu = 0.1$.



Solution

a) Since we neglect the masses of the cable and the pulley, and the pulley is frictionless, the magnitudes of tension T at both ends of the cable are the same.

Load m_2 accelerates downward with magnitude a. The load m_1 connected by the string to the first load moves at the same acceleration to the right. The equations according to the Newton's 2nd Law for each load are

$$\begin{cases} m_1\vec{a} = \vec{T} + \vec{N} + m_1\vec{g}, \\ m_2\vec{a} = \vec{T} + m_2\vec{g}. \end{cases}$$

The projections of these equations on the x and y axes are

$$\begin{cases} m_1 a = T, \\ 0 = m_1 g - N, \\ m_2 a = m_2 g - T. \end{cases}$$

Substituting the tension T from the first equation into the third equation gives

$$m_2 a = m_2 g - m_1 a \, .$$

It follows that

$$a = \frac{m_2 g}{m_1 + m_2} = \frac{3 \cdot 9.8}{2+3} = 5.88 \text{ m/s}^2.$$

b) If the friction is between the first load and the surface of the table, the equations of motion are given by

$$\begin{cases} m_1 \vec{a} = \vec{T} + \vec{N} + m_1 \vec{g} + \vec{F}_{fr}, \\ m_2 \vec{a} = \vec{T} + m_2 \vec{g}. \end{cases}$$

As a consequence, the projections on the x and y axes are

$$\begin{cases} m_1 a = T - F_{fr}, \\ 0 = m_1 g - N, \\ m_2 a = m_2 g - T. \end{cases}$$

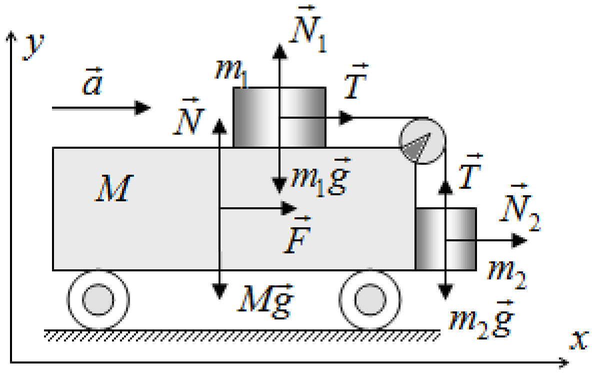
Allow for the fact that $F_{fr} = \mu \cdot N = \mu \cdot m_1 g$, we obtain

$$m_2 a = m_2 g - m_1 a - F_{fr} = m_2 g - m_1 a - \mu \cdot m_1 g \, ,$$

$$a = \frac{(m_2 - k \cdot m_1) \cdot g}{m_1 + m_2} = \frac{(3 - 0.1 \cdot 2) \cdot 9.8}{2+3} = 5.49 \, \text{m/s}^2.$$

Problem 2.23

What horizontal force must be applied to the cart $M = 5$ N in order that the blocks ($m_1 = 2$ kg, $m_2 = 3$ kg) remain stationary relative to the cart? Assume all surfaces, wheels, and pulley are frictionless.



Solution

If the cart is at rest the system of two blocks must move with acceleration (see the Problem 2.22). When the cart moves at certain acceleration the blocks may be at rest respectively the cart.

The equations of the motion of two loads according to the Newton's 2nd Law are

$$\begin{cases} m_1\vec{a} = \vec{T} + \vec{N}_1 + m_1\vec{g}, \\ m_2\vec{a} = \vec{T} + \vec{N}_2 + m_2\vec{g}. \end{cases}$$

As we examine the case of stationary state of the blocks let's find x-projection of the first equation and y-projection of the second equation:

$$\begin{cases} x: \quad m_1 a = T, \\ y: 0 = T - m_2 g, \end{cases}$$

$$\begin{cases} m_1 a = T, \\ m_2 g = T. \end{cases}$$

Hence, $m_1 a = m_2 g$ and

$$a = \frac{m_2 g}{m_1}.$$

The cart with two blocks is moving due to the force $\vec{F}$ action.

$$F = \left(m_1 + m_2 + M\right) \cdot a = \left(m_1 + m_2 + M\right) \cdot \frac{m_2 g}{m_1} == \frac{(2+3+5)\cdot 3 \cdot 9.8}{2} = 147 \text{ N}.$$

Thus, if the applied force is 147 N, the loads are at rest relatively to the cart.

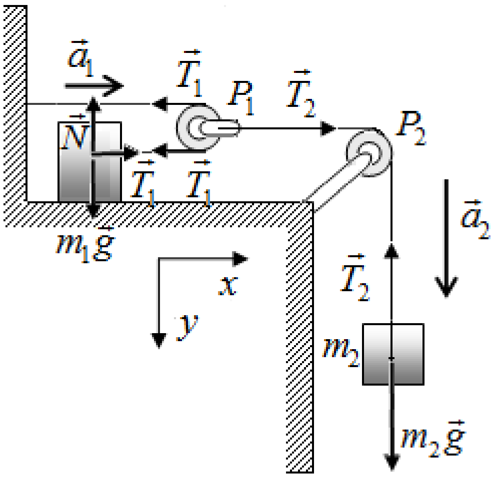
Problem 2.24

The load of mass m_1 on the frictionless horizontal surface is connected to the load of mass m_2 through a very light pulley P_1 and a light fixed pulley P_2 as shown in the Figure. a) If a_1 and a_2 are the accelerations of m_1 and m_2, respectively, what is the relation between these accelerations? Find b) the accelerations a_1 and a_2, and c) the tensions in the strings.

Solution

a) The load m_1 moves twice the distance P_1 moves in the same time, m_1 has twice the acceleration of P_1: $a_1 = 2a_2$.

b) Using Newton's 2nd Law and the relationship between the tensions, the equations of the load motion are



$$\begin{cases} m_1\vec{a}_1 = \vec{T} + m_1\vec{g} + \vec{N}, \\ m_2\vec{a}_2 = m_2\vec{g} + \vec{T}_2, \\ \vec{T}_2 + 2\vec{T}_1 = 0. \end{cases}$$

Allow for the fact that $a_1 = 2a_2$, the projections of these equations on the x and y axes are

$$\begin{cases} T_1 = m_1 a_1 = 2m_1 a_2 \\ 0 = m_1 g - N, \\ m_2 a_2 = m_2 g - T_2, \\ T_2 = 2T_1. \end{cases}$$

Then the third equation of the system is

$$m_2 a_2 = m_2 g - T_2 = m_2 g - 2T_1 = m_2 g - 2(2m_1 a_2).$$

$$m_2 a_2 + 4m_1 a_2 = m_2 g,$$

$$a_2 = \frac{m_2 g}{4m_1 + m_2}.$$

The acceleration of the first load is

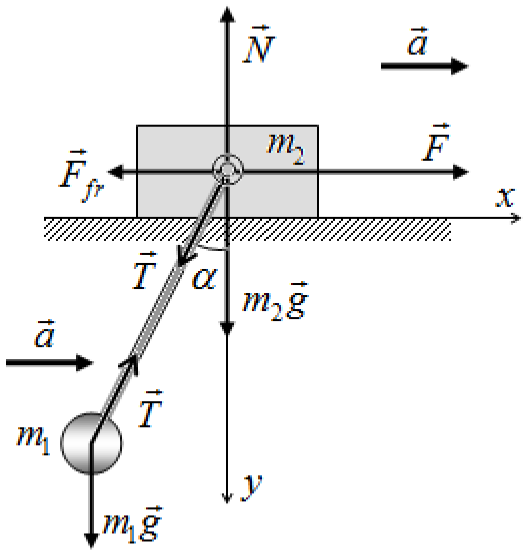
$$a_1 = 2a_2 = \frac{2m_2 g}{4m_1 + m_2}.$$

c) The tensions of the strings are

$$T_1 = \frac{2m_1 m_2 g}{4m_1 + m_2}, \qquad T_2 = 2T_1 = \frac{4m_1 m_2 g}{4m_1 + m_2}.$$

Problem 2.25

The load of mass $m_1 = 5$ kg by means of weightless rod and frictionless joint is attached to the mass $m_2 = 30$ kg. The coefficient of kinetic friction between m_2 and the horizontal surface is $\mu = 0.2$. The constant force F causes the system to accelerate. The angle $\alpha = 20°$ is constant. Determine the acceleration of the block a, tension T and applied force F.



Solution

The equations of motion using Newton's 2nd Law for both loads are

$$\begin{cases} m_1\vec{a} = m_1\vec{g} + \vec{T}, \\ m_2\vec{a} = m_2\vec{g} + \vec{T} + \vec{F} + \vec{F}_{fr} + \vec{N}. \end{cases}$$

The projections of these equations on the chosen x and y axes are following

$$x: \quad m_1 a = T\sin\alpha, \tag{1}$$

$$y: \quad 0 = m_1 g - T\cos\alpha, \tag{2}$$

$$x: \quad m_2 a = F - T\sin\alpha - F_{fr}, \tag{3}$$

$$y: \quad 0 = m_2 g - N + T\cos\alpha. \tag{4}$$

Dividing equation (1) by (2), we obtain

$$\frac{m_1 a}{m_1 g} = \frac{T\cdot\sin\alpha}{T\cdot\cos\alpha},$$

$$a = g\frac{\sin\alpha}{\cos\alpha} = g\cdot\tan\alpha = 9.8\cdot\tan 20^0 = 3.57 \text{ m/s}^2.$$

Tension from the equation (1) is

$$T = \frac{m_1 a}{\sin\alpha} = \frac{5\cdot 3.57}{\sin 20^0} = 52.2 \text{ N}.$$

Normal force from the equation (4) is given as

$$N = m_2 g + T\cos\alpha,$$

as a result, the friction force is

$$F_{fr} = \mu \cdot N = \mu \cdot (m_2 g + T\cos\alpha) = 0.2\left(30 \cdot 9.8 + 52.2 \cdot \cos 20^0\right) = 68.6 \text{ N}.$$

Finally, the applied force using the equation (3) is

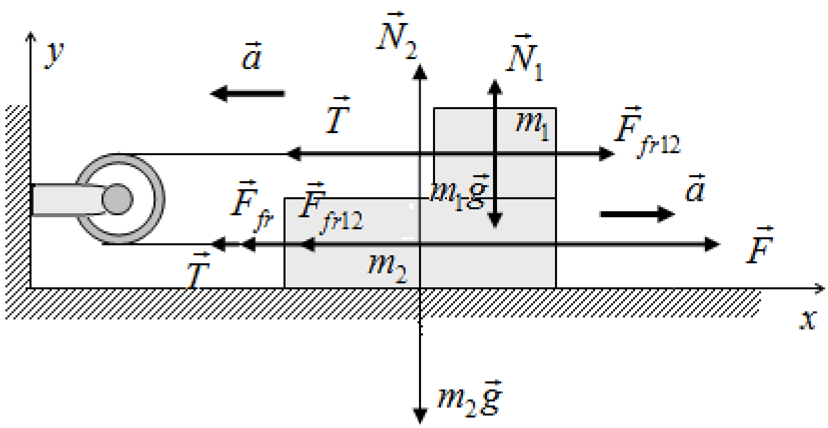
$$F = m_2 a + T\sin\alpha + F_{fr} = 30 \cdot 3.57 + 52.2 \cdot \sin 20^0 + 68.6 = 193.6 \text{ N}.$$

Problem 2.26

A block of mass $m_1 = 3$ *kg sits on top of a block* $m_2 = 5$ *kg, which is on a horizontal surface. The second block is pulled to the right with a force* $\vec{F}$ *as shown in the figure. The coefficient of static friction between all surfaces is* $\mu_1 = 0.65$ *and the kinetic coefficient is* $\mu_2 = 0.42$. *a) What is the minimum value of the force* F *needed to move two blocks? b) If the force is 10% greater than your answer for (a), what is the acceleration of each block?*

Solution

The forces that are acting on the blocks are: the top block (m_1): the gravity $m_1\vec{g}$, the normal force on contact between two blocks $\vec{N}_1$, friction force between two blocks F_{fr12}, the tension $\vec{T}$; the bottom block (m_2): the gravity $m_2\vec{g}$, the normal force on contact between the block and the surface $\vec{N}_2$, friction force between two blocks F_{fr12}, friction force between the block and the surface $\vec{F}_{fr}$, the tension $\vec{T}$.



Equations of the blocks motion (projections on the horizontal and vertical axes):

for the top block

$$\begin{cases} m_1 a = T - F_{fr12}, \\ 0 = -m_1 g + N_1; \end{cases}$$

for the bottom block

$$\begin{cases} m_2 a = F - T - F_{fr12} - F_{fr}, \\ 0 = -m_1 g - m_2 g + N_2. \end{cases}$$

The normal forces calculated using above equations are

$$N_1 = m_1 g\,,$$

$$N_2 = m_1 g + m_2 g = \left(m_1 + m_2\right) g\,.$$

a) If two blocks are just to move, the force of static friction is at its maximum, and so the frictions forces are as follows

$$F_{fr12} = \mu_1 \cdot N_1 = \mu_1 \cdot m_1 g\,,$$

$$F_{fr} = \mu_1 N_2 = \mu_1 \cdot \left(m_1 + m_2\right) g\,.$$

The acceleration $a = 0$, therefore, the first equations of each system are

$$\begin{cases} 0 = T - F_{fr12}, \\ 0 = F - T - F_{fr12} - F_{fr}, \end{cases}$$

$$\begin{cases} T = \mu_1 \cdot m_1 g, \\ F = T + F_{fr12} + F_{fr}, \end{cases}$$

Substitute $T = \mu_1 \cdot m_1 g$ into the second equation, and find the desired force

$$F = \mu_1 \cdot m_1 g + \mu_1 \cdot m_1 g + \mu_1 \cdot \left(m_1 + m_2\right) g =$$
$$= \mu_1 g\left(3m_1 + m_2\right) = 0.65 \cdot 9.8\left(3 \cdot 3 + 5\right) = 89.18 \quad \text{N.}$$

b) Now the applied force $F_1 = 1.1 \cdot F = 1.1 \cdot 89.18 = 98.1$ N, and the loads move at acceleration $\vec{a}$. The horizontal projections of equation of motion are

$$\begin{cases} m_1 a = T - F_{fr12}, \\ m_2 a = F - T - F_{fr12} - F_{fr}. \end{cases}$$

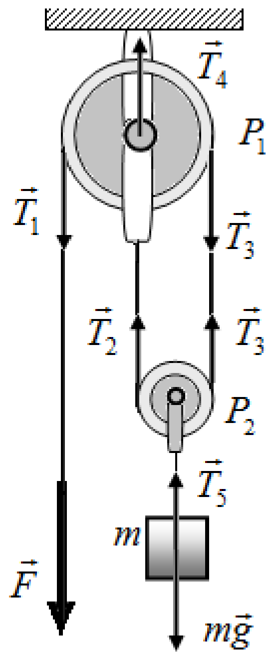
Adding the equations gives

$$(m_1 + m_2)a = F_1 - 2F_{fr12} - F_{fr} = F_1 - \mu_2 g(3m_1 + m_2),$$

$$a = \frac{F_1 - \mu_2 g(3m_1 + m_2)}{m_1 + m_2} = \frac{98.1 - 0.42 \cdot 9.8 \cdot (3 \cdot 3 + 5)}{3 + 5} = 5.06 \text{ m/s}^2$$

Problem 2.27

An object of mass m is held in place by an applied force $\vec{F}$ and a pulley system as shown in Figure. The pulleys are massless and frictionless. Find the tension in each section of rope, $\vec{T}_1$, $\vec{T}_2$, $\vec{T}_3$, $\vec{T}_4$, $\vec{T}_5$ and the magnitude of $\vec{F}$. Determine the mechanical advantage of a pulley system.



Solution

A pulley is a simple machine made with a rope wrapped around a wheel. A pulley changes the direction of the force, making it easier to lift things to high-rise areas. In the problem we have a combination of fixed and movable pulleys which is created to provide mechanical advantage and a change of direction for effort force.

A fixed pulley P_1, which wheel does not move, changes the direction of the effort force. It does not increase the size of the effort force. A movable pulley P_2 moves with the load. It allows the effort to be less than the weight of the load. The main advantage of combined pulley is that the amount of effort is less than half of the load. The main disadvantage is it travels a very long distance.

As it is seen from the scheme,

$$F = T_1,$$

$$mg = T_5.$$

For the bottom pulley P_2

$$T_5 = T_2 + T_3,$$

and for the top pulley P_1

$$T_4 = T_1 + T_2 + T_3.$$

Since the pulleys are not starting to rotate and are frictionless

$$T_1 = T_3, \text{ and } T_2 = T_3,$$

Therefore,

$$T_5 = 2T_2, \text{ and } T_2 = \frac{T_5}{2} = \frac{mg}{2} .$$

Then,

$$T_1 = T_2 = T_3 = \frac{mg}{2},\ T_4 = \frac{3mg}{2},\ T_5 = mg .$$

The applied force is

$$F = T_1 = \frac{mg}{2}.$$

This result shows that this pulley system gives the mechanical advantage

$$MA = \frac{mg}{F} = \frac{mg \cdot 2}{mg} = 2 .$$

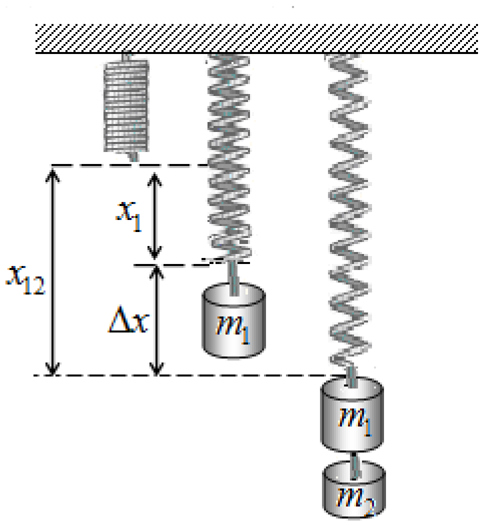
Problem 2.28

A block of $m_1 = 2$ kg is suspended from the ceiling trough a massless spring of spring constant $k = 100$ N/m. What is the elongation of the spring? If another block of mass $m_2 = 1$ kg is added to the first block what would be the further elongation?

Solution

When the first block is at equilibrium state, the gravity of the block is equal to the elastic force of the spring

$$m_1 g = kx_1,$$



the elongation of the spring is equal to

$$x_1 = \frac{m_1 g}{k} = \frac{2 \cdot 9.8}{100} = 0.196 \text{ m.}$$

If another block of mass m_2=1 kg is added,

$$(m_1 + m_2) g = k x_{12},$$

the elongation is

$$x_{12} = \frac{(m_1 + m_2) g}{k} = \frac{(2+1) \cdot 9.8}{100} = 0.294 \text{ m.}$$

Additional elongation of the spring is

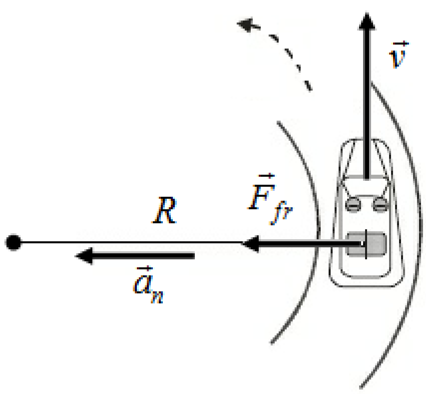
$$\Delta x = x_{12} - x_1 = 0.294 - 0.196 = 0.098 \text{ m.}$$

Problem 2.29

A car of the mass m = 1000 *kg moves along the circular path on a flat, horizontal road. If the radius of the curve is* 40 *m and the coefficient of static friction between the tires and dry pavement is* $\mu = 0.55$, *find the maximum speed the car can have and still make the turn successfully.*

Solution

The force that enables the car to remain in the circular way is the force of static friction. This is the static friction, because no slipping occurs at the point of contact between road and tires. If this force of static friction were zero, for example, if the car were on an icy road, the car would continue in a straight line and slide off the road. The maximum speed the car can have around the curve is the speed at which it is on the verge of skidding outward.



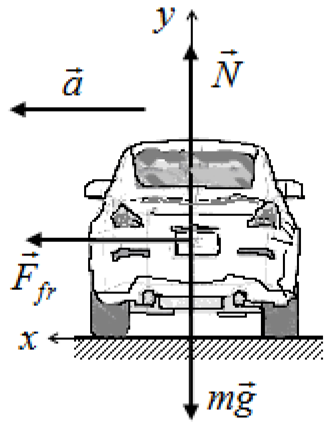
According to the Newton's 2nd Law the equation of the car's motion is

$$m\vec{a} = m\vec{g} + \vec{N} + \vec{F}_{fr}, \qquad (1)$$

and x and y projections of (1) are

$$x: \; ma = F_{fr}, \qquad (2)$$

$$y: \quad 0 = N - mg.$$



The maximum friction force is

$$F_{fr} = \mu \cdot N = \mu \cdot mg.$$

Since the car moves along the curvilinear path its normal (or centripetal) acceleration is $a = a_n = v^2/R$, and the equation (2) is given by

$$\frac{mv^2}{R} = \mu \cdot mg.$$

The maximum speed of the car is determined by the maximum friction force is equal to

$$v = \sqrt{\mu \cdot R \cdot g} = \sqrt{0.55 \cdot 40 \cdot 9.8} = 14.7 \text{ m/s} = 52.9 \text{ km/h}.$$

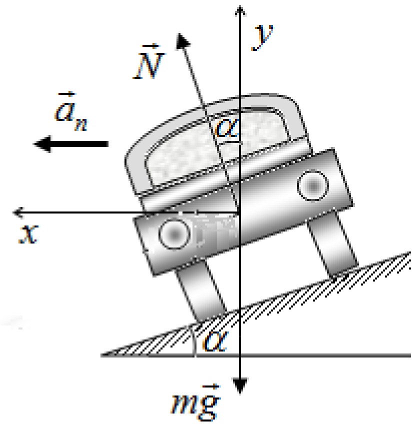
Problem 2.30

The designated speed for the ramp is to be 13.4 *m/s; the radius of the curve is* 40 *m. At what angle should the curve be banked?*

Solution

The curved parts of the roads have to be designed in such a way that a car will not have to rely on friction to round the curve without skidding. In other words, a car moving at the designated speed can negotiate the curve even when the road is covered with ice. Such a ramp is usually banked; this means the roadway is tilted toward the inside of the curve.

On a level (unbanked) road the force that causes the centripetal acceleration is the force of static friction between car and road, as we saw in the Problem 2.29. However, if the road is banked at an angle α, the equation of the motion according Newton's 2nd Law is



$$m\vec{a} = m\vec{g} + \vec{N},$$

Its x and y projections are

$$\begin{cases} ma = N\sin\alpha, \\ 0 = N\cos\alpha - mg. \end{cases}$$

Now, the normal force $\vec{N}$ has a horizontal component $N\sin\alpha$, pointing toward the center of the curve. Because the ramp is to be designed so that the force of static friction is zero, only the component $N\sin\alpha$ causes the centripetal acceleration.

$$m\frac{v^2}{R} = N\sin\alpha = \frac{mg\cdot\sin\alpha}{\cos\alpha} = mg\cdot\tan\alpha,$$

$$\tan\alpha = \frac{v^2}{gR}.$$

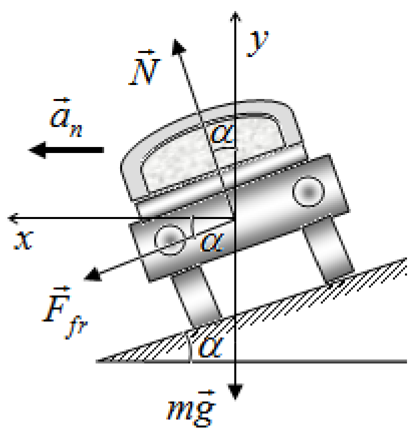
$$\alpha = \arctan\frac{v^2}{gR} = \arctan\frac{17^2}{9.8\cdot 40} = \arctan 0.74 = 36.4°.$$

Problem 2.31

The freeway off-ramp is circular with 60-m radius. The off-ramp has a slope $\alpha = 15°$. *If the coefficient of static friction between the tires of a car and the road is* $\mu = 0.55$, *what is the maximum speed at which it can enter the ramp without losing traction?*

Solution

The equation that describes the motion of the car according to the Newton's 2nd Law is



$$m\vec{a} = m\vec{g} + \vec{N} + \vec{F}_{fr},$$

and its x and y projections are

$$\begin{cases} ma = N\cdot\sin\alpha + F_{fr}\cos\alpha, \\ 0 = N\cdot\cos\alpha - mg - F_{fr}\cdot\sin\alpha. \end{cases}$$

Since the friction force is $F_{fr} = \mu \cdot N$, these equations take the form

$$\begin{cases} ma = N\sin\alpha + \mu N\cos\alpha, \\ mg = N\cos\alpha - \mu N\sin\alpha, \end{cases}$$

$$\begin{cases} ma = N(\sin\alpha + \mu\cos\alpha), \\ mg = N(\cos\alpha - \mu\sin\alpha). \end{cases}$$

Dividing the first equation of the system by the second one gives

$$\frac{\cancel{m}a}{\cancel{m}g} = \frac{\cancel{N}(\sin\alpha + \mu\cos\alpha)}{\cancel{N}(\cos\alpha - \mu\sin\alpha)},$$

Finally, taking into account that $a = a_n = \dfrac{v^2}{R}$, we obtain

$$v = \sqrt{gR \cdot \left(\frac{\sin\alpha + \mu \cdot \cos\alpha}{\cos\alpha - \mu \cdot \sin\alpha}\right)}.$$

Substitution of the numerical values gives the maximum speed

$$v = \sqrt{\frac{9.8 \cdot 60 \cdot (\sin 15° + 0.55 \cdot \cos 15°)}{\cos 15° - 0.55 \cdot \sin 15°}} = 23.9 \text{ m/s} = 85.9 \text{ km/h}.$$

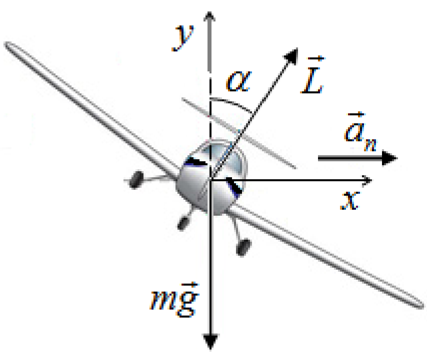
Problem 2.32

A jet of mass $m = 2 \cdot 10^5$ kg, flying at 120 m/s, banks to make a horizontal circular turn. The radius of the turn is 3500 m. Calculate the necessary lifting force.

Solution

The equation of the jet motion is

$$m\vec{a} = m\vec{g} + \vec{L}.$$



When the aircraft banks the lift force may be broken into horizontal and vertical components. The horizontal component causes centripetal acceleration acting toward the center of the circular path. The projections of Newton's 2nd Law equation on the chosen x and y axes are

$$\begin{cases} ma = L\sin\alpha, \\ 0 = L\cos\alpha - mg. \end{cases}$$

Since the jet is moving along the circular way its acceleration is

$$a = a_n = \frac{v^2}{R}.$$

Then

$$\begin{cases} m\dfrac{v^2}{R} = L\sin\alpha, \\ mg = L\cos\alpha. \end{cases}$$

Dividing the first equation by the second one gives

$$\tan\alpha = \frac{v^2}{gR}.$$

The banking angle is

$$\alpha = \arctan\left(\frac{v^2}{g\cdot R}\right) = \arctan\left(\frac{120^2}{9.8\cdot 3500}\right) = \arctan 0.42 = 22.8^\circ.$$

The lifting force from the second equation of the system is

$$L = \frac{mg}{\cos\alpha} = \frac{2\cdot 10^5\cdot 9.8}{\cos 22.8^\circ} = 2.13\cdot 10^6\ \text{N}.$$

Problem 2.33

The car is moving along the convex bridge with radius of curvature R =100 *m at the speed* v =36 *km/h. a) Find the force of its pressure on the middle of the bridge? b) Find the force of pressure on concave bridge of the same radius of curvature.*

Solution

Since the car moves along the curvilinear path it has normal (centripetal) acceleration

$$a = a_n = \frac{v^2}{R}.$$

Newton's 2nd Law gives the following equation of the car's motion

$$m\vec{a} = m\vec{g} + \vec{N}.$$

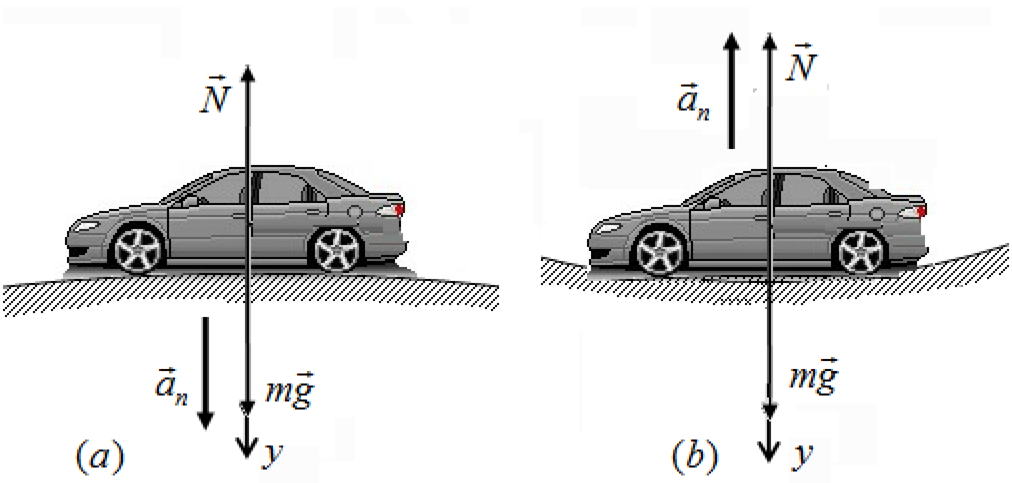
According to Newton's 3rd Law, the magnitude of the force of car's pressure on the bridge is equal to the magnitude of the normal force *N*.

If the *y*-axis is directed downwards, the projections of the equation are:

a) for the car's motion with the speed $v = 36$ km/h = $\frac{36000}{3600} = 10$ m/s across the convex bridge

$$ma_n = mg - N,$$

$$F = N = m(g - a_n) = m\left(g - \frac{v^2}{R}\right) = 5 \cdot 10^3\left(9.8 - \frac{10^2}{100}\right) = 4.4 \cdot 10^4 \text{ N}.$$



b) for the motion across the concave bridge

$$-ma_n = mg - N,$$

$$F = N = m(g + a_n) = m\left(g + \frac{v^2}{R}\right) = 5 \cdot 10^3\left(9.8 + \frac{10^2}{100}\right) = 5.4 \cdot 10^4 \text{ N}.$$

Problem 2.34

A car traveling at the speed v = 30 m/s. The coefficient of kinetic friction between the tyres and the road is $\mu = 0.8$. The instantaneous radius of curvature of the car's path is R = 200 m. If the driver applies the brakes and the car's wheels lock, what is the resulting deceleration of the car tangent to its path, if the car is a) at the top of a hill; b) at the bottom of a depression?

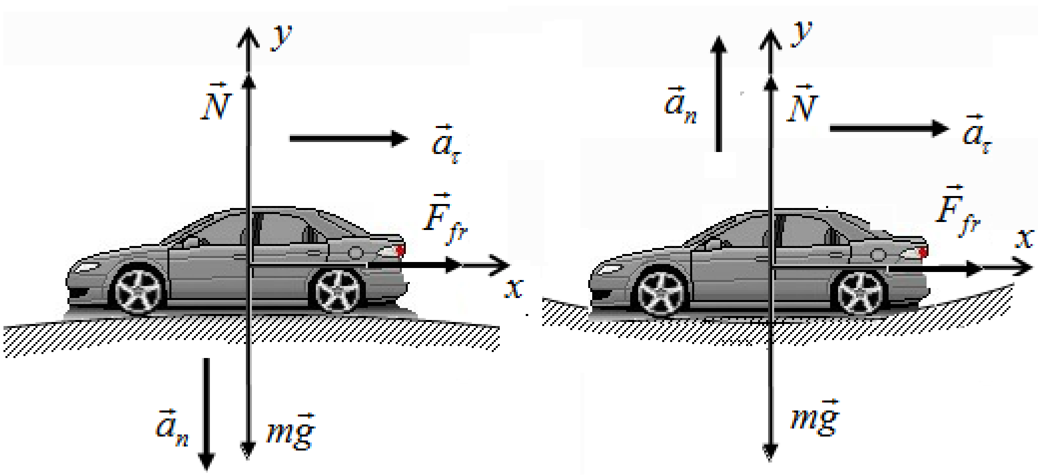
Solution

The car undergoing circular motion is continuously subjected to the acceleration, called normal (or centripetal) acceleration, directed toward the center of the curvature. The equation of the car's motion is

$$m\vec{a} = m\vec{g} + \vec{N} + \vec{F}_{fr}.$$

Depending on the type of the road (convex or concave) the projections on the x and y axes are

$$\begin{cases} ma_\tau = F_{fr}, \\ \pm ma_n = N - mg. \end{cases}$$



a) If the car is at the top of a hill

$$\begin{cases} ma_\tau = F_{fr}, \\ -ma_n = N - mg. \end{cases}$$

The normal force from the second equation may be expressed as

$$N = m(g - a_n) = m\left(g - \frac{v^2}{R}\right).$$

Substitution N into the first equation gives

$$a_\tau = \frac{F_{fr}}{m} = \frac{kN}{m} = \frac{\mu \cdot \cancel{m}}{\cancel{m}}\left(g - \frac{v^2}{R}\right) = \mu\left(g - \frac{v^2}{R}\right) = 0.8\left(9.8 - \frac{30^2}{200}\right) = 4.24 \text{ m/s}^2.$$

b) For the car at the bottom of a depression the similar calculations give

$$\begin{cases} ma_\tau = F_{fr}, \\ ma_n = N - mg. \end{cases}$$

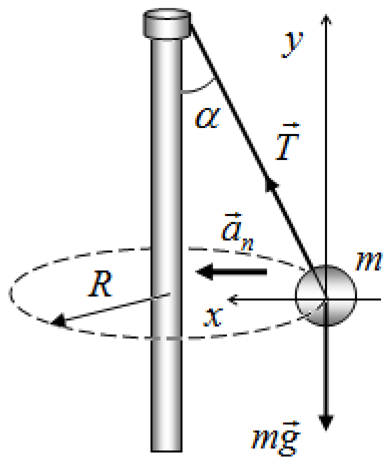
The normal force is $N = m(g + a_n) = m\left(g + \frac{v^2}{R}\right)$, and the magnitude of car's deceleration is

$$a_\tau = \frac{F_{fr}}{m} = \frac{\mu N}{m} = \frac{\mu \cdot \cancel{m}}{\cancel{m}}\left(g + \frac{v^2}{R}\right) = \mu\left(g + \frac{v^2}{R}\right) = 0.8\left(9.8 + \frac{30^2}{200}\right) = 11.44 \text{ m/s}^2.$$

It is important to note that in both cases the acceleration is directed opposite to the velocity of the car.

Problem 2.35

The ball of mass $m = 1$ kg rotates around the vertical pole in a horizontal circular path. The angle $\alpha = 30°$ and the length of the string is $L = 1.22$ m.
a) What is the magnitude of the ball's velocity?
b) Find the velocity and the angle at which the string will break if the tensile strength of the string is $T_0 = 50$ N.



Solution

a) The equation of the ball motion according to Newton's 2nd Law is

$$m\vec{a} = m\vec{g} + \vec{T}.$$

Projections of this equation on the x and y axes are

$$\begin{cases} ma = T\sin\alpha, \\ 0 = T\cos\alpha - mg. \end{cases}$$

Since the ball moves along the curvilinear path it has normal (or centripetal) acceleration

$$a_n = \frac{v^2}{R}.$$

Dividing the first equation of the system by the second one we obtain

$$\frac{ma_n}{mg} = \frac{T\sin\alpha}{T\cos\alpha},$$

$$\frac{a_n}{g} = \frac{\sin\alpha}{\cos\alpha} = \tan\alpha .$$

The normal acceleration of the ball is

$$a_n = \frac{v^2}{R} = \frac{v^2}{L\sin\alpha}.$$

The velocity of the ball is

$$v = \sqrt{Lg \cdot \sin\alpha \cdot \tan\alpha} = \sqrt{1.22 \cdot 9.8 \cdot \sin 30^\circ \cdot \tan 30^\circ} = 1.86 \text{ m/s}.$$

b) When the string makes the angle β with vertical and the tension of the string is critical, the projections on the equation of the ball's motion are

$$\begin{cases} ma_{n1} = T_0\sin\beta, \\ 0 = T_0\cos\beta - mg. \end{cases}$$

The second equation gives

$$\cos\beta = \frac{mg}{T_0},$$

therefore,

$$\beta = \arccos\left(\frac{mg}{T_0}\right) = \arccos\left(\frac{1 \cdot 9.8}{50}\right) = \arccos 0.196 = 78.7° .$$

The first equation of the system is

$$ma_{n1} = \frac{mv_0^2}{R_1} = T_0 \sin\beta .$$

Taking into account that $R_1 = L\sin\beta$, we obtain

$$v_0 = \sin\beta \cdot \sqrt{\frac{T_0 L}{m}} = \sin 78.7° \cdot \sqrt{\frac{50 \cdot 1.22}{1}} = 7.66 \text{ m/s.}$$

Problem 2.36

A ball is attached to one end of a rigid massless rod, while an identical ball is attached to the center of the rod. Each ball has a mass $m = m_1 = m_2 = 0.3$ kg, and the length of each half of the rod is $R = 0.6$ m. This arrangement is held by the empty end and is whirled around in a horizontal circle at a constant rate, so each ball is in uniform circular motion. Ball m_1 travels at a constant speed of $v_1 = 4$ m/s. Find the tension in each half of the rod.

Solution

Two balls are moving at same angular speed

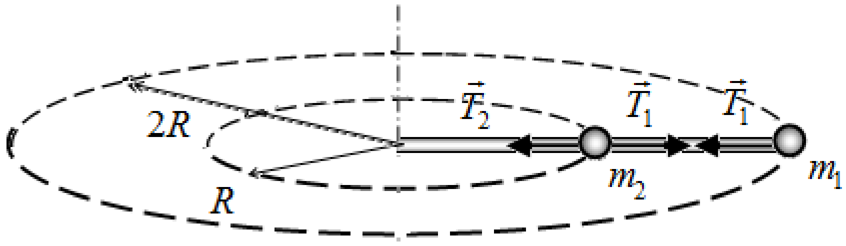
$$\omega = \frac{v_1}{2R} = \frac{v_2}{R},$$

therefore, the linear speed of the second ball may be expressed as

$$v_2 = \frac{v_1 \cdot R}{2R} = \frac{v_1}{2} = \frac{4}{2} = 2 \text{ m/s.}$$

The horizontal projections of the equations according to the Newton's 2nd Law are

$$\begin{cases} m_1 a_1 = T_1, \\ m_2 a_2 = T_2 - T_1. \end{cases},$$



The accelerations of the balls moving along the circular paths are normal (or centripetal) accelerations $a_1 = \frac{v_1^2}{2R}$ and $a_2 = \frac{v_2^2}{R}$. Then the equations of motion are

$$\begin{cases} m_1 \cdot \frac{v_1^2}{2R} = T_1, \\ m_2 \cdot \frac{v_2^2}{R} = T_2 - T_1. \end{cases}$$

The tension in the right half of the rod follows from the first of these equations

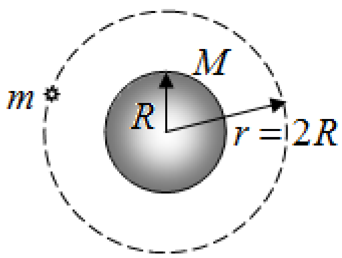
$$T_1 = m_1 \cdot \frac{v_1^2}{2R} = \frac{0.3 \cdot 4^2}{2 \cdot 0.6} = 4 \text{ N}.$$

The tension in the second half is

$$T_2 = T_1 + m_2 \cdot \frac{v_2^2}{R} = 4 + \frac{0.3 \cdot 2^2}{0.6} = 6 \text{ N}.$$

Problem 2.37

A satellite of mass 300 *kg is in a circular orbit around the Earth* ($R = 6.37 \cdot 10^6$ *m,* $M = 5.98 \cdot 10^{24}$ *kg*) *at an altitude equal to the Earth's mean radius Find a) the gravitational force acting on it, b) the satellite's orbital speed, and c) the period of its revolution.*



Solution

a) The radius of the satellite's orbit is equal to twice the radius of the Earth. Since the mean radius of the Earth is $6.37 \cdot 10^6$ m, the orbit radius is

$$r = 2R = 2 \cdot 6.37 \cdot 10^6 = 1.27 \cdot 10^7 \text{ m}.$$

The satellite is always at this distance from the center of the Earth; Newton's law of gravitation tells us the force which the Earth exerts on the satellite

$$F = \gamma \frac{m \cdot M}{r^2} = 6.67 \cdot 10^{-11} \frac{300 \cdot 5.98 \cdot 10^{24}}{\left(1.27 \cdot 10^7\right)^2} = 742 \text{ N}.$$

This force is always directed toward the center of the Earth and it is the only force which acts on the satellite during the motion along the orbit.

b) According to Newton's 2nd Law, the equation of the satellite motion is

$$m\vec{a} = \vec{F}\,.$$

Since the satellite rotate about the Earth it has centripetal acceleration $a = \dfrac{v^2}{r}$. As a result,

$$\frac{\cancel{m}v^2}{r} = \gamma \frac{\cancel{m} \cdot M}{r^2}\,.$$

So the satellite's orbital speed is

$$v = \sqrt{\frac{\gamma M}{r}} = \sqrt{\frac{6.67 \cdot 10^{-11} \cdot 5.98 \cdot 10^{24}}{1.27 \cdot 10^7}} = 5.6 \cdot 10^3 \text{ m/s}.$$

c) Recall that the speed of an object in uniform circular motion is related to the period T and radius r by

$$v = \frac{2\pi r}{T}.$$

Hence the period of the satellite's orbit is

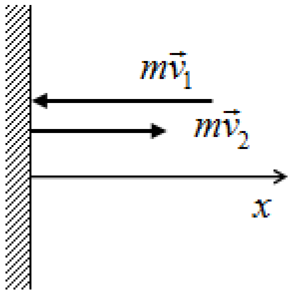
$$T = \frac{2\pi r}{v} = \frac{2\pi \cdot 1.27 \cdot 10^7}{5.6 \cdot 10^3} = 1.42 \cdot 10^4 \text{ s} = 3.96 \text{ hr}.$$

Problem 2.38

A steel ball of mass $m = 10$ g moving at a speed $v = 100$ m/s along the normal to the wall hits it and bounces at the same speed. Find the linear momentum obtained by the wall.

Solution

Assume that the collision of the ball and the wall is perfectly elastic. According to the law of conservation of linear momentum for the "ball – wall" closed (isolated) system the magnitude of the linear momentum obtained by the wall (p_w) is equal to the magnitude of ball's linear momentum increment(Δp_b).



$$\vec{p}_w = \Delta\vec{p}_b = \vec{p}_2 - \vec{p}_1 = m\vec{v}_2 - m\vec{v}_1.$$

The projection of this equation on x- axis is

$$p_w = mv_2 - (-mv_1) = mv_2 + mv_1.$$

Since $v_1 = v_2 = v$, therefore,

$$p_w = 2mv = 2 \cdot 10^{-2} \cdot 10^2 = 2 \text{ kg·m/s}.$$

Problem 2.39

What is the impulse of a force of 10 *N acting on a ball for* 2 *seconds? A* 2 *kg-ball is initially at rest. What is the velocity of the ball after the force has acted on it?*

Solution

The definition of impulse is force over a time, so we have to do a simple calculation:

$$F\Delta t = 10 \cdot 2 = 20 \text{ N·s}.$$

Recall that an impulse causes a change in linear momentum. Because the particle starts with zero velocity, its initial momentum is equal to zero. Thus,

$$F = \frac{\Delta p}{\Delta t} = \frac{\Delta(mv)}{\Delta t} = m\frac{\Delta v}{\Delta t},$$

$$F\Delta t = m\Delta v = mv_2 - mv_1 = mv_2,$$

$$v_2 = \frac{F\Delta t}{m} = \frac{20}{2} = 10 \text{ m/s}.$$

The ball's final velocity is 10 m/s. This problem is the simplest form of the impulse–momentum theorem.

Problem 2.40

A 3-kg particle has a velocity of $\left(3\vec{i}-4\vec{j}\right)$ *m/s. Find x and y components of its linear momentum and the magnitude of its total momentum.*

Solution

Using the definition of momentum and given data of mass m and velocity gives

$$\vec{p}=m\vec{v}=3\left(3\vec{i}-4\vec{j}\right)=\left(9\vec{i}-12\vec{j}\right)\ \text{kg·m/s}.$$

This means that the components of the momentum of the particle are

$$p_x=+9\ \text{kg·m/s},\quad \text{and}\quad p_y=-12\ \text{kg·m/s}.$$

The magnitude of its momentum is

$$p=\sqrt{p_x^2+p_y^2}=\sqrt{9^2+\left(-12\right)^2}=15\ \text{kg·m/s}.$$

Problem 2.41

A child bounces a ball on the sidewalk. The linear impulse delivered by the sidewalk is 2 *N·s during the* $1.25\cdot10^{-3}$ *s of contact. What is the magnitude of the average force exerted on the ball by the sidewalk.*

Solution

The magnitude of the change in momentum of the ball (*or* impulse delivered to the ball) is $|\Delta p|=2$ N·s. The direction of the impulse is upward, since the initial momentum of the ball was downward and the final momentum is upward.

Since the time over which the force was acting was $\Delta t=1.25\cdot10^{-3}$ s then from the definition of average force we get:

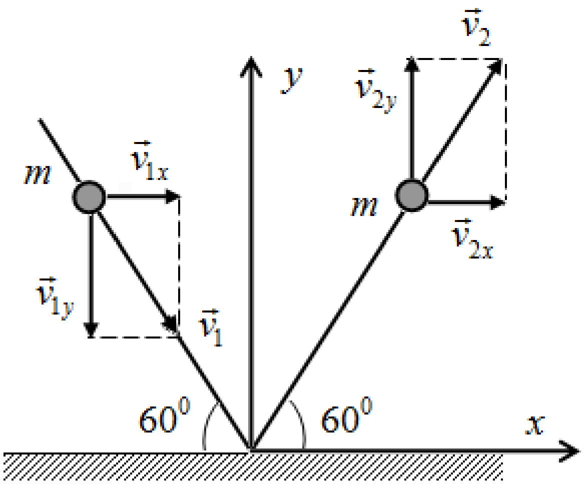
$$F=\frac{|\Delta p|}{\Delta t}=\frac{2}{1.25\cdot10^{-3}}=1600\ \text{N}.$$

Problem 2.42

A 3-*kg steel ball strikes a wall with a speed of* 10 *m/s at an angle of* 60° *with the surface. It bounces off with the same speed and angle, as shown in figure. If the ball is in contact with the wall for* 0.2 *s, what is the average force exerted on the wall by the ball.*

Solution

The average force is defined as $\vec{F} = \frac{d\vec{p}}{dt}$, so firstly we have to find the change in momentum of the ball. Since the ball has the same speed before and after bouncing from the wall, it is clear that its x-component of velocity (see the figure) stays the same $v_{x1} = v_{x2}$ and so the x-component of linear momentum stays the same $p_{x1} = p_{x2}$. But the y-component of momentum does change. The initial y-component of velocity is



$$v_{1y} = -10 \cdot \sin 60° = -8.7 \text{ m/s},$$

and the final y-component of velocity is

$$v_{2y} = 10 \cdot \sin 60° = 8.7 \text{ m/s}.$$

So the change in y-component of momentum is

$$\Delta\vec{p}_y = m\vec{v}_{2y} - m\vec{v}_{1y}$$

$$\Delta p_y = mv_{2y} - mv_{1y} = m\left(v_{2y} - v_{1y}\right) = 3\left[8.7 - \left(-8.7\right)\right] = 52 \,\text{kg·m/s}.$$

The average y-component of the force on the ball is

$$F = \frac{\Delta p_y}{\Delta t} = \frac{52}{0.2} = 260\,\text{N}.$$

Since F has no x-component, the average force has magnitude 260 N and points in the y-direction (away from the wall).

Problem 2.43

A 2 kg-ball is thrown straight up into the air with an initial velocity of 10 *m/s. Using the impulse-momentum theorem, calculate the time of flight of the ball* (g =10 *m/s²*).

Solution

Once the ball is thrown up, it is acted on by a constant force $m\vec{g}$. This force causes a change in momentum until the ball has reversed directions, and lands with the velocity of 10 m/s. Thus we can calculate the total change in momentum:

$$\Delta p = p_2 - p_1 = mv_2 - mv_1 = 2 \cdot (10) - 2(-10) = 40 \text{ kg·m/s}.$$

Now we turn to the impulse–momentum theorem to find the time of flight:

$$F\Delta t = mg\Delta t = \Delta p .$$

Thus, taking $g = 10$ m/s²,

$$\Delta t = \frac{\Delta p}{mg} = \frac{40}{2 \cdot 10} = 2 \text{ s}.$$

The ball has a time of flight of 2 seconds.

This result may be obtained based on kinematics. The height of the ball is

$$h = \frac{v_0^2}{2g} = \frac{100}{2 \cdot 10} = 5 \text{ m}.$$

The velocity at the top point is $v = 0$, therefore, $v = v_0 - gt$ gives

$$0 = v_0 - gt \text{ and } v_0 = gt .$$

Then $h = v_0 t - \dfrac{gt^2}{2} = gt^2 - \dfrac{gt^2}{2} = \dfrac{gt^2}{2}$.

Finally

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \cdot 5}{10}} = 1 \text{ s}.$$

The time of upward motion is equal to the time of downward motion; it follows that the time of flight is 2 s. We obtain same result. But the calculation using impulse-momentum theorem was much easier than the one using kinematic equations.

Problem 2.44

Machine gun fires 35 g bullets at a speed of 750 *m/s. If the gun can fire* 200 *bullets/min, what is the average force the shooter must exert to keep the gun from moving?*

Solution

The gun interacts with the bullets; it exerts a brief, strong force on each of the bullets which in turn exerts an "equal and opposite" force on the gun. The gun's force changes the bullet's momentum from zero $p_1 = mv_1 = 0$ (as they are initially at rest) to the final value of

$$p = p_2 - p_1 = p_2 = mv_2 = 0.035 \cdot 750 = 26.2 \text{ kg·m/s}.$$

This is also the change in momentum for each bullet: $p = \Delta p$.

Now, since 200 bullets are fired every minute (60 s), we should count the interaction time as the time to fire one bullet,

$$\Delta t = \frac{60}{200} = 0.3 \text{ s},$$

because every 0.3 s, a firing occurs again and the average force that we compute will be valid for an interval of time for which many bullets are fired. So the average force of the gun on the bullets is

$$F = \frac{\Delta p}{\Delta t} = \frac{26.2}{0.3} = 87.5 \text{ N}.$$

From Newton's 3rd Law, there must be an average backwards force of the bullets on the gun of magnitude 87.5 N. If there were no other forces acting on the gun, it would accelerate backward. To keep the gun in place, the shooter must exert a force of 87.5 N in the forward direction.

We can also work with the numbers as follows. In one minute, 200 bullets were fired, and a total momentum of

$$p_{\text{total}} = 200 \cdot 26.2 = 5.24 \cdot 10^3 \text{ kg·m/s}$$

was imparted to them. So during this time period (60 seconds) the average force on the whole set of bullets was

$$F = \frac{\Delta p_{\text{total}}}{\Delta t} = \frac{5.24 \cdot 10^3}{60} = 87.5 \text{ N}.$$

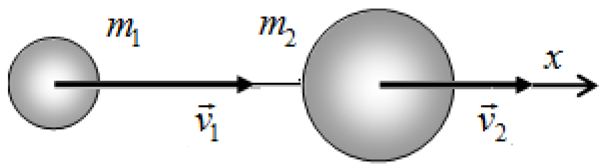
As before, this is also the average backwards force of the bullets on the gun and the force required to keep the gun in place.

Problem 2.45

A 2 kg ball (m_1) with a velocity of v_1 = 3 m/s collides head-on in an elastic manner with an 8 kg ball (m_2) with a velocity v_2 = 1 m/s. What are the velocities after the collision?

Solution

If the head-on (central) elastic collision takes place the laws of conservation of linear momentum and energy are carried out:



$$\begin{cases} m_1\vec{v}_1 + m_2\vec{v}_2 = m_1\vec{u}_1 + m_2\vec{u}_2, \\ \dfrac{m_1\vec{v}_1^2}{2} + \dfrac{m_2\vec{v}_2^2}{2} = \dfrac{m_1\vec{u}_1^2}{2} + \dfrac{m_2\vec{u}_2^2}{2}, \end{cases}$$

$$\begin{cases} m_1(\vec{v}_1 - \vec{u}_1) = m_2(\vec{u}_2 - \vec{v}_2), \\ \left[m_1(\vec{v}_1 - \vec{u}_1)\right](\vec{v}_1 + \vec{u}_1) = \left[m_2(\vec{u}_2 - \vec{v}_2)\right](\vec{u}_2 + \vec{v}_2) \end{cases}$$

Therefore,

$$\vec{v}_1 + \vec{u}_1 = \vec{u}_2 + \vec{v}_2.$$

After two mathematical operations: multiplication of $\vec{v}_1 + \vec{u}_1 = \vec{u}_2 + \vec{v}_2$ by m_2 and subtraction of the product from $m_1(\vec{v}_1 - \vec{u}_1) = m_2(\vec{u}_2 - \vec{v}_2)$; and multiplication of $\vec{v}_1 + \vec{u}_1 = \vec{u}_2 + \vec{v}_2$ by m_1 and addition of the result to $m_1(\vec{v}_1 - \vec{u}_1) = m_2(\vec{u}_2 - \vec{v}_2)$, we obtain velocities of the balls after collision:

$$\vec{u}_1 = \frac{2m_2\vec{v}_2 + (m_1 - m_2)\vec{v}_1}{m_1 + m_2},$$

$$\vec{u}_2 = \frac{2m_1\vec{v}_1 + (m_2 - m_1)\vec{v}_2}{m_1 + m_2}.$$

The projection of the velocities on x–axis and substitution of the data gives:

$$u_1 = \frac{2 \cdot 8 \cdot 1 + (2-8) \cdot 3}{2+8} = -0,2 \text{ m/s},$$

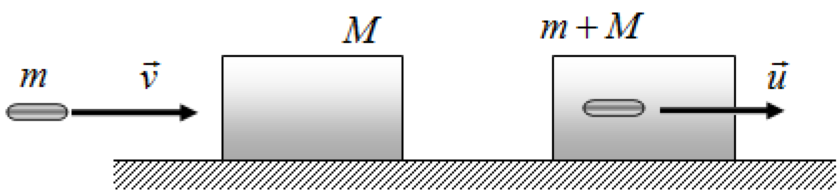
$$u_2 = \frac{2 \cdot 2 \cdot 3 + (8-2) \cdot 1}{2+8} = 1,8 \text{ m/s}.$$

The sign "minus" in the first expression means that due to collision the first ball moves in opposite direction, meanwhile the second ball moves in its previous direction.

Problem 2.46

A 10 *g bullet is stopped in a block of wood of mass m* = 5 *kg. The speed of the "bullet-wood" combination immediately after the collision is* 0.6 *m/s. What was the original speed of the bullet?*

Solution



The scheme of the collision just before and after the bullet embeds itself in the wood is given in the figure below. The collision (and embedding of the bullet) takes place very rapidly; for that brief time the bullet and block essentially form an isolated system because any external forces will be of no importance compared to the enormous forces between the bullet and the block. So the total momentum of the system will be conserved; it is the same before and after the collision.

Just before the collision, only the bullet with mass m is in motion and its velocity is $\vec{v}$. So the initial momentum is $\vec{p} = m\vec{v}$. Just after the collision, the "bullet-block" combination, with its mass of $(m+M)$ has a velocity $\vec{u}$. So the final momentum is

$$\vec{p}\,' = (m+M)\vec{u}\,.$$

In this problem there is only motion along the horizontal axis, so we need the condition that the total momentum along this axis is conserved.

$$\vec{p} = \vec{p}'.$$

Consequently,

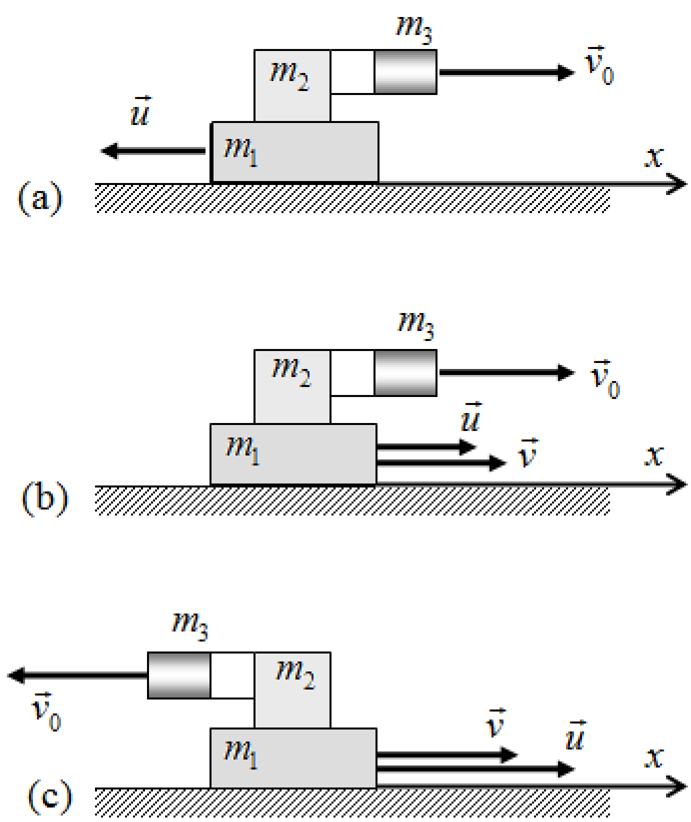
$$mv = (m + M)u.$$

The speed of the bullet was

$$v = \frac{(m+M)u}{m} = \frac{(10^{-2}+5)\cdot 0.6}{10^{-2}} = 301 \text{ m/s.}$$

Problem 2.47

The platform of mass m_1 =10 *tonnes is on the rails. The gun of mass* m_2 =5 *tonnes fixed on the platform shoot by the shell of mass* m_3 =100 *kg. The shot is directed along the rails. Find the velocity of the platform with the gun immediately after the shot if the velocity of the shell is* v_0 =500 *m/s respectively the gun. Solve the problem if the platform with gun on it at the instant of firing a) was at rest; b) moved at the velocity* v = 18 *km/h and the shot was fired in the direction of its motion; and c) moved at the velocity* v = 18 *km/h and the shot was fired in the opposite direction.*



Solution

The solution of this problem is based on the law of conservation of linear momentum for the closed system consisting of the platform $(\vec{p}_1)$, the gun $(\vec{p}_2)$, and the shell $(\vec{p}_3)$. The linear momentum of the system is vector sum of the linear momenta of the objects entering into the examined system.

a) The linear momentum of the system before the shot (the platform was at rest $v = 0$) is

$$\vec{p} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 = m_1\vec{v} + m_2\vec{v} + m_3\vec{v} = (m_1 + m_2 + m_3)\vec{v} = 0 .$$

The linear momentum of the system after the shot is

$$\vec{p}' = \vec{p}_1' + \vec{p}_2' + \vec{p}_3' = m_1\vec{u} + m_2\vec{u} + m_3\vec{v}_0 = (m_1 + m_2)\vec{u} + m_3\vec{v}_0 .$$

According the law of conservation of linear momentum $\vec{p} = \vec{p}'$, consequently,

$$0 = (m_1 + m_2)\vec{u} + m_3\vec{v}_0 .$$

Assuming that the direction of the motion of the platform with the gun ($\vec{u}$) after firing is opposite to the direction the shell motion ($\vec{v}_0$) of the shell, we have chosen the sign “minus” for this velocity. Hence, the projections of the equation on x-axis is

$$0 = -(m_1 + m_2)u + m_3 v_0 ,$$

$$(m_1 + m_2)u = m_3 v_0 ,$$

$$u = \frac{m_3 v_0}{m_1 + m_2} = \frac{100 \cdot 500}{10000 + 5000} = 3.33 \text{ m/s.}$$

If we are can’t assume the direction of the motion after collision, we can choose it positive (the velocity directed along the positive direction of x-axis), and calculation shows the correctness of our choice.

b) If the platform is moving at the speed $v = 18$ km/h = 5 m/s, the linear momentum of the system before shooting is not zero, and the law of conservation of linear momentum is

$$(m_1 + m_2 + m_3)\vec{v} = \left(m_1 + m_2\right)\vec{u} + m_3\left(\vec{v}_0 + \vec{v}\right).$$

Projecting the equation on x-axis gives

$$(m_1 + m_2 + m_3)v = -\left(m_1 + m_2\right)u + m_3\left(v_0 + v\right).$$

The speed of the platform with the gun is

$$u = \frac{m_3\left(v_0 + v\right) - (m_1 + m_2 + m_3)v}{m_1 + m_2} =$$

$$= \frac{10^2 \cdot (500+5) - (10^5 + 5000 + 10^2)\cdot 5}{10^5 + 5000} = -1{,}67 \text{ m/s}.$$

We supposed that the platform after shooting was moving in the opposite direction relatively its previous direction, but the negative result of calculation shows that it was moving in the same direction but at smaller speed.

c) The law of the conservation of linear momentum is the same as in the second case, but the projection on x-axis is

$$(m_1 + m_2 + m_3)v = \left(m_1 + m_2\right)u + m_3\left(-v_0 + v\right).$$

The calculation of the speed of the platform with the gun gives

$$u = \frac{-m_3\left(-v_0 + v\right) + (m_1 + m_2 + m_3)v}{m_1 + m_2} =$$

$$\frac{-10^2(-500+5) + (10^5 + 5000 + 10^2)\cdot 5}{10^5 + 5000} = -8.33 \text{ m/s}.$$

This result shows that the platform is moving in the same direction as before shooting.

Problem 2.48

The man of mass m_1=60 *kg running at the velocity* v_1=2 *m/s jumps onto the carriage of mass* m_2=80 *kg moving at the velocity* v_2=1 *m/s. Find the velocity of the carriage with the man if a) the man catches up the carriage; and b) the man moves towards the carriage.*

Solution

The law of the conservation of linear momentum is

$$m_1\vec{v}_1 + m_2\vec{v}_2 = \left(m_1 + m_2\right)\vec{u}$$

a) If the man catches up the carriage, the initial velocities of both objects are of the same directions, therefore, the projection of the equation on x-axis is

$$m_1 v_1 + m_2 v_2 = \left(m_1 + m_2\right)u\ .$$

The velocity of the carriage with man is

$$u = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} = \frac{60 \cdot 2 + 80 \cdot 1}{60 + 80} = 1.43 \text{ m/s.}$$

b) If the man moves towards the carriage, the velocities of the objects have the different signs. The projection of the equation of the law of conservation of linear momentum is

$$m_1 v_1 - m_2 v_2 = (m_1 + m_2) u .$$

The velocity of the carriage with man is

$$u = \frac{m_1 v_1 - m_2 v_2}{m_1 + m_2} = \frac{60 \cdot 2 - 80 \cdot 1}{60 + 80} = 0.29 \text{ m/s.}$$

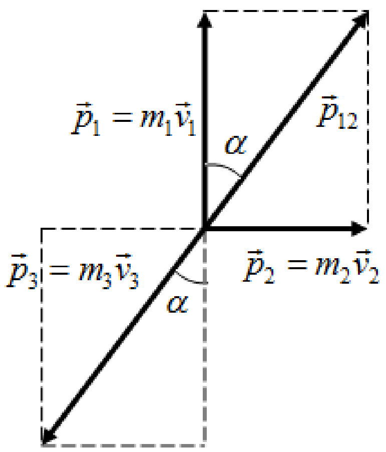
The carriage is moving in the direction of previous motion of the man.

Problem 2.49

The shell that has been shot upwards blew up at the highest point of its trajectory. The first fragment of mass m_1 *= 1 kg has the horizontal velocity* v_1 *= 400 m/s. The second fragment of mass* m_2 *= 1.5 kg has the vertical velocity* v_2 *= 200 m/s. Find the velocity of the third fragment of mass* m_3 *= 2 kg.*

Solution

As the shell blew up in the highest point of trajectory where its velocity was zero, the linear momentum of the shell before the explosion was zero too. According to the law of conservation of linear momentum the vector sum of the momenta of all fragments after the burst has to be zero as well. Therefore,



$$0 = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 .$$

The linear momenta of the first and the second fragments are

$$p_1 = m_1 v_1 = 1 \cdot 400 = 400 \text{ kg·m/s,}$$

$$p_2 = m_2 v_2 = 1.5 \cdot 200 = 300 \text{ kg·m/s.}$$

As the figure shows, the vector sum of $\vec{p}_1$ and $\vec{p}_2$ is vector $\vec{p}_{12}$, which magnitude is

$$p_{12} = \sqrt{p_1^2 + p_2^2} = \sqrt{400^2 + 300^2} = 500 \text{ kg·m/s}.$$

To make the vector sum zero, the linear momentum of the third fragment $\vec{p}_3$ has to be equal in value and opposite in sign to the linear momentum $\vec{p}_{12}$: $p_3 = p_{12} = 500$ kg·m/s.

Since $p_3 = m_3 v_3 = p_{12}$, the speed of the third fragment is

$$v_3 = \frac{p_{12}}{m_3} = \frac{500}{2} = 250 \text{ kg·m/s}.$$

The angular momentum $\vec{p}_3$ makes the angle α with downward vertical direction

$$\alpha = \arctan\left(\frac{p_2}{p_1}\right) = \arctan\left(\frac{300}{400}\right) = 36.9°.$$

Problem 2.50

A particle moves under the action of the constant force $\vec{F} = \left(5\vec{i} - 2\vec{j}\right)N$. *If its displacement is* $6\vec{j}$ *m, what is the work done by this force?*

Solution

The work done by a constant force is the dot product of this force and the displacement caused by this force.

$$A = \left(\vec{F}, \vec{s}\right) = \left(5\vec{i} - 2\vec{j}\right)\left(6\vec{j}\right) = -12 \text{ J}.$$

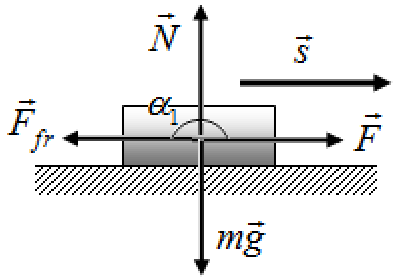
Problem 2.51

A 40 *kg box initially at rest is pushed* 5 *m along a rough horizontal floor with a constant applied horizontal force of* 130 *N. If the coefficient of friction between the box and floor is* $\mu = 0.3$, *determine a) the work done by the applied force, b) the energy lost due to friction, c) the change in kinetic energy of the box, and d) the final speed of the box.*

Solution

a) The motion of the box and the forces which do work on it are shown in the figure below. The constant applied force $\vec{F}$ points in the same direction as the displacement $\vec{s}$. The expression for the work done by a constant force gives

$$A=\left(\vec{F},\vec{s}\right)=F\cdot s\cdot\cos\alpha=130\cdot 5\cdot\cos 0^0=650\ \text{J}.$$



The applied force does $650\,\text{J}$ of work.

b) Figure shows all forces acting on the box.

The vertical forces acting on the box are gravity ($m\vec{g}$, downward) and the floor's normal force ($\vec{N}$, upward). It follows that $N=mg$ and so the magnitude of the friction force is

$$F_{fr}=\mu N=\mu mg=0.3\cdot 40\cdot 9.8=120\ \text{N}.$$

The friction force is directed opposite the direction of motion ($\alpha_1=180°$) and so the work that it does is

$$A_{fr}=\left(\vec{F}_{fr},s\right)=F_{fr}s\cdot\cos\alpha_1=F_{fr}\cdot s\cdot\cos 180^0=1.2\cdot 10^2\cdot 5\cdot(-1)=-590\ \text{J},$$

or we might say that $590\,\text{J}$ is *lost* to friction.

c) Since the normal force and gravity are at right angles to the displacement these forces do no work on the box as it moves. The net work done is

$$A_{net}=A+A_{fr}=650-590=60\ \text{J}.$$

According to Work-Kinetic Energy Theorem, the net work is equal to the change in kinetic energy of the box.

$$\Delta W_k=W_{k2}-W_{k1}=A_{net}=60\ \text{J}.$$

d) Here, the initial kinetic energy W_{k1} was zero because the box was initially at rest. So the final kinetic energy is $W_{k2}=60\,\text{J}$.

From the definition of kinetic energy $W_k=\dfrac{mv^2}{2}$, we can get the final speed of the box which is equal to

$$v=\sqrt{\frac{2W_k}{m}}=\sqrt{\frac{2\cdot 60}{40}}=\sqrt{3}=1.73\ \text{m/s}.$$

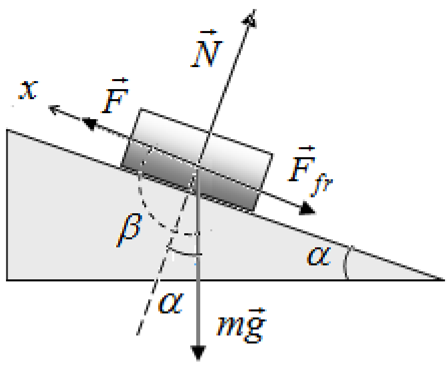
Problem 2.52

A crate of mass 10 *kg is pulled up a rough incline with an initial speed of* 1.5 *m/s. The pulling force is* 100 *N parallel to the incline, which makes an angle of* $\alpha = 20°$ *with the horizontal. The coefficient of kinetic friction is* $\mu = 0.4$*, and the crate is pulled* 5 *m. a) How much work is done by gravity? b) How much energy is lost due to friction? c) How much work is done by the* 100 *N force? d) What is the change in kinetic energy of the crate? e) What is the speed of the crate after being pulled* 5 *m.*

Solution

a) The work done by gravity using the definition of work is $A_{gr} = \left(m\vec{g}, \vec{d}\right)$. The magnitude of the gravity force is $mg = 10 \cdot 9.8 = 98\,\text{N}$ and the magnitude of displacement is 5 m. As it seen from the figure, the angle that the gravity makes with the direction of displacement (along the positive x-axis) is

$$\beta = 90° + \alpha = 90° + 20° = 110°.$$



The work done by gravity is

$$A_{gr} = mg \cdot d \cdot \cos\beta = 98 \cdot 5 \cdot \cos 110° = -168 \text{ J}.$$

b) To find the work done by friction, we need to know the friction force. The forces on the block are shown in the figure. The normal force between the slope and the block is $N = mg \cdot \cos\alpha = mg \cdot \cos 20°$ so as to cancel the normal component of the force of gravity. Then the force of kinetic friction on the block that points downward along the slope (opposite the motion) is

$$F_{fr} = \mu \cdot N = \mu \cdot mg \cdot \cos 20° = 0.4 \cdot 10 \cdot 9.8 \cdot 0.94 = 36.8 \text{ N}.$$

This force points exactly opposite the direction of the displacement $\vec{d}$, so the work done by friction is

$$A_{fr} = F_{fr} \cdot d \cdot \cos 180^0 = 36.8 \cdot 5 \cdot (-1) = -184 \text{ J}.$$

c) Applied force $F = 100$ N directed upward along the slope in the same direction as the displacement $\vec{d}$. So the work that is does is

$$A_{appl} = \left(\vec{F}, \vec{d}\right) = F \cdot d \cdot \cos 0° = 100 \cdot 5 \cdot 1 = 500 \text{ J}.$$

d) We have now found all works done by each of the forces acting on the crate as it moved: gravity, friction and the applied force. (We should note the normal force of the surface also acted on the crate, but being perpendicular to the motion, it did no work.) The net work done was:

$$A_{net} = A_{gr} + A_{fr} + A_{appl} = -168 - 184 - 500 = 148 \text{ J}.$$

From the Work-Kinetic Energy Theorem, this net work is equal to the change in kinetic energy of the box: $\Delta W_k = A_{net} = 148$ J.

e) The initial kinetic energy of the crate was $W_{k1} = \dfrac{mv_1^2}{2} = \dfrac{10 \cdot 1.5^2}{2} = 11.2$ J.

If the final speed of the crate is v_2, then the change in kinetic energy was:

$$\Delta W_k = W_{k2} - W_{k1} = \frac{mv_2^2}{2} - W_{k1}.$$

Using our answers from previous parts $\Delta W_k = A_{net} = 148$ J and $W_{k1} = 11.2$ J, we get $W_{k2} = \Delta W_k + W_{k1} = 148 + 11.2 = 159.2$ J.

$$v_2 = \sqrt{\frac{2W_{k2}}{m}} = \sqrt{\frac{2 \cdot 159.2}{10}} = 5.64 \text{ m/s}.$$

The final speed of the crate is 5.64 m/s.

Problem 2.53

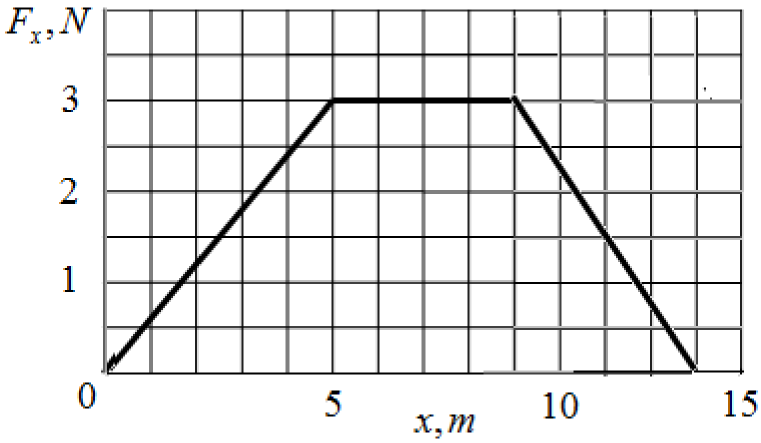
A particle is subject to a force F_x that varies with position. Find the work done by the force on the body as it moves a) from x =0 to x =5 m, b) from x = 5 m to x = 10 m and c) from x = 10 m to x = 14 m. d) What is the total work done by the force over the distance x = 0 to x = 14 m?

Solution

Since the force depends on time, the work done when a particle moves along a straight line

$$W = \int_{x_1}^{x_2} F_x dx.$$

Geometrically this is just the "area under the curve" of $F_x(x)$ from x_1 to x_2.



a) The figure (from $x_1 = 0$ to $x_2 = 5$ m) is the triangle which "area" is equal to the half of a rectangle of base 5 m and height 3 N. So the work done is

$$W_1 = \frac{3 \cdot 5}{2} = 7.5 \text{ J.}$$

It is important that when we evaluate the "area", we just keep the units which go along with the base and the height; here they were meters and Newtons, the product of which is a Joule.

So the work done by the force for this displacement is 7.5 J.

b) The region under the curve from $x_2 = 5$ m to $x_3 = 10$ m is a full rectangle of base 5 m and height 3 N. The work done for this movement is

$$W_2 = 3 \cdot 5 = 15 \text{ J.}$$

c) For the movement from $x_3 = 10$ m to $x_4 = 14$ m, the region under the curve is a half rectangle of base 5 m and height 3 N. The work done is

$$W_3 = \frac{3 \cdot 4}{2} = 6 \text{ J.}$$

d) The total work done over the distance $x_1 = 0$ to $x_4 = 14$ m is the sum of the three separate "areas",

$$W = W_1 + W_2 + W_3 = 7.5 + 15 + 6 = 28.5 \text{ J.}$$

Problem 2.54

What work is done by a force $\vec{F}=\left(2x\vec{i}+3\vec{j}\right)N$*, with* x *in meters, that moves a particle from position* $\vec{r}_1=\left(2\vec{i}+3\vec{j}\right)m$ *to position* $\vec{r}_2=\left(-4\vec{i}-3\vec{j}\right)m$*?*

Solution

According to the general definition of work (for a two-dimensional problem),

$$W=\int_{x_1}^{x_2}F_x\left(\vec{r}\right)dx+\int_{y_1}^{y_2}F_y\left(\vec{r}\right)dy.$$

If $F_x=2x$ and $F_y=3$ (we mean that the force is in Newtons when x is in meters, and the work will come out in Joules), we obtain

$$W=\int_2^{-4}2x\,dx+\int_3^{-3}3\,dy=x^2\Big|_2^{-4}+3x\Big|_3^{-3}=(16-4)+(-9-9)=-6\text{ J}.$$

Problem 2.55

The sledge after motion down from the hill of height $h_1=10$ *m and slope angle* $\alpha=30°$*, has covered* 3 *m of horizontal way, and begins to move upwards along the incline with angle* $\beta=45°$*. If the coefficient of friction doesn't vary along all sectors of motion and equals* $\mu=0.1$*, find the height* h_2 *of the sledge ascent along the hill.*

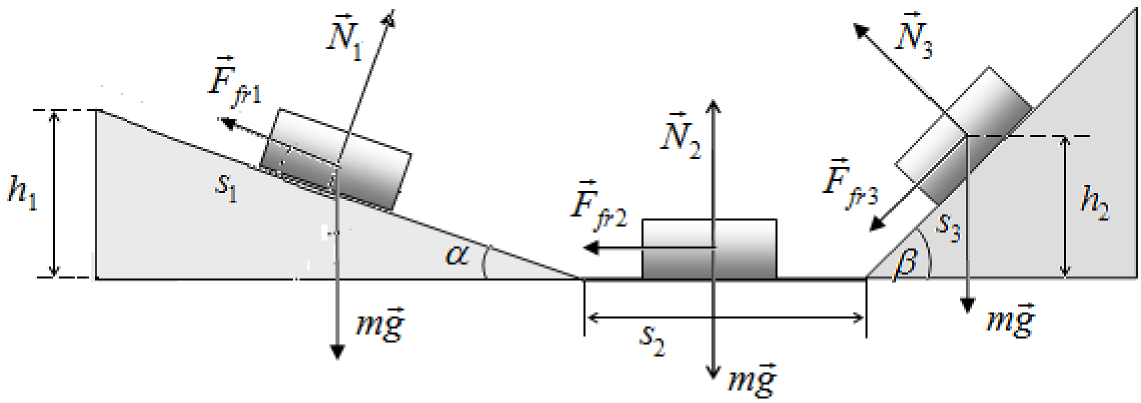
Solution

According to the law of conservation of energy the potential energy of the sledge $\left(W_p\right)$ at its initial point is expended in kinetic energy $\left(W_{k1}\right)$ and the work $\left(A_{fr1}\right)$ done by friction force at the motion downward along the first incline.

$$W_p=W_{k1}+A_{fr1}.$$

Due to the kinetic energy $\left(W_{k1}\right)$ the sledge has a possibility of motion along the horizontal way, at that, a part of this energy is expended in the work $\left(A_{fr2}\right)$ of friction force

$$W_{k1}=A_{fr2}+W_{k2}\ .$$



And finally, the kinetic energy W_{k2} allows the ascent of the sledge to the height h_2, with that,

$$W_{k2} = A_{fr_3} + W_{p2}.$$

Combining all previous equations, we obtain

$$W_p = A_{fr1} + A_{fr2} + A_{fr_3} + W_{p2}.$$

Friction force at the first segment of the distance is equal to $F_{fr1} = \mu \cdot N_1 = \mu \cdot mg \cdot \cos\alpha$, at the second segment: $F_{fr2} = \mu \cdot N_2 = \mu \cdot mg$, and at the third segment: $F_{fr3} = \mu \cdot N_3 = \mu \cdot mg \cdot \cos\beta$. Accordingly, the works of friction forces are:

$$A_{fr1} = \mu \cdot mg \cdot \cos\alpha \cdot s_1,\ A_{fr2} = \mu \cdot mg \cdot s_2, \text{ and } A_{fr3} = \mu \cdot mg \cdot \cos\beta \cdot s_3.$$

Taking into account that

$$s_1 = \frac{h_1}{\sin\alpha},\ s_2 = 3\,\text{m, and}\, s_3 = \frac{h_2}{\sin\beta},$$

we write down the equation

$$mgh_1 = \mu \cdot mg \cdot \cos\alpha \cdot s_1 + \mu \cdot mg \cdot s_2 + \mu \cdot mg \cdot \cos\beta + mg \cdot h_2,$$

$$h_1 = \mu\left(\cos\alpha \cdot s_1 + s_2 + \cos\beta \cdot s_3\right) + h_2,$$

$$\frac{h_1}{\mu} = \cos\alpha \cdot \frac{h_1}{\sin\alpha} + s_2 + \cos\beta \cdot \frac{h_2}{\sin\beta} + \frac{h_2}{\mu} = \frac{h_1}{\tan\alpha} + s_2 + \frac{h_2}{\tan\beta} + \frac{h_2}{\mu},$$

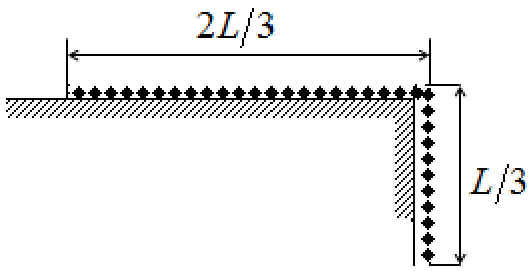
$$\frac{h_1}{\mu} - \frac{h_1}{\tan\alpha} - s_2 = \frac{h_2}{\tan\beta} + \frac{h_2}{\mu},$$

$$h_2 = \frac{h_1(\tan\alpha - \mu) - \mu \cdot s_2 \cdot \tan\alpha}{\tan\beta + \mu} \cdot \frac{\tan\beta}{\tan\alpha} =$$

$$= \frac{10(0.577 - 0.1) - 0.1 \cdot 3 \cdot 0.577}{0.1 + 1} \cdot \frac{1}{0.577} = 7.24\,\text{m}.$$

Problem 2.56

A uniform chain of length L *and mass* m *overhangs a horizontal table with its* $2/3$ *part on the table. The friction coefficient between the table and the chain is* μ*. Find the work done by friction during the period slips the table.*



Solution

Let us consider an element dx of the chain on the table distance x from the edge of the table. Its mass is $\left(\frac{m}{L}\right)dx$, and the normal force that the table exerts on it is $N = \left(\frac{m}{L}\right)dx \cdot g$. The friction force is $F_{fr} = \mu N = \mu\left(\frac{m}{L}\right)dx \cdot g$. As this element is made to fall off the table the work done by the friction force is

$$dA = -\mu \cdot \left(\frac{m}{L}\right) \cdot dx \cdot g \cdot x\ .$$

Since two thirds part of the chain is on the table the integration for finding the work to be done by the friction force is between 0 and $2L/3$, and noting that the friction force and displacement are in opposite direction we have

$$A = -\frac{\mu \cdot mg}{L}\int_0^{2L/3} x \cdot dx = -\frac{\mu \cdot mg}{L} \cdot \frac{x^2}{2}\Bigg|_0^{2L/3} = -\frac{\mu \cdot mg}{L} \cdot \frac{4L^2}{9} = -\frac{\mu \cdot mg \cdot L}{9}.$$

Problem 2.57

A particle has linear momentum of 10 *kg·m/s and a kinetic energy of* 25 *J. What is the mass of the particle?*

Solution

The linear momentum of the particle is $p = mv$. The speed may be found as

$$v = \frac{p}{m}.$$

The kinetic energy of the particle is

$$W_k = \frac{mv^2}{2} = \frac{m}{2}\left(\frac{p}{m}\right)^2 = \frac{p^2}{2m}.$$

From our knowledge of energy and momentum, we can state the mass of the ball from these two quantities.

$$m = \frac{p^2}{2W_k} = \frac{100}{2 \cdot 25} = 2 \text{ kg}.$$

This method of finding the mass of a particle is commonly used in particle physics, when particles decay too quickly to be massed, but when their momentum and energy can be measured.

Problem 2.58

A block of mass m_1 = 3 kg is released from A located at the height H =10 m and moved along a frictionless track. It makes a head-on elastic collision with a block of mass m_2 = 6 kg at B, initially at rest. Calculate a) the velocities of the blocks after the elastic collision, and b) the maximum height h to which block m_1 rises after collision.

Solution

a) At the top of the slop the block m_1 had potential energy $W_{p1} = m_1 gH$.

We can use the fact that energy is conserved as m_1 slides down the smooth (frictionless) slope. The potential energy is changed to kinetic energy $W_{k1} = \frac{m_1 v_1^2}{2}$ when it reaches the bottom.

Conservation of energy gives $W_{p1} = W_{k1}$ or $m_1 gH = \dfrac{m_1 v_1^2}{2}$, so that

$$v_1 = \sqrt{2gH} = \sqrt{2 \cdot 9.8 \cdot 10} = 14 \text{ m/s}.$$

We assume that the velocity of m_1 just before striking v_1 is positive since this block obviously moving forward at the bottom of the slope. Then it makes an elastic (one-dimensional) collision with m_2 at point B. The velocities of the objects after their elastic collision may be found from

$$\begin{cases} \vec{u}_1 = \dfrac{2m_2\vec{v}_2 + (m_1 - m_2)\vec{v}_1}{m_1 + m_2}, \\ \vec{u}_2 = \dfrac{2m_1\vec{v}_1 + (m_2 - m_1)\vec{v}_2}{m_1 + m_2}. \end{cases}$$

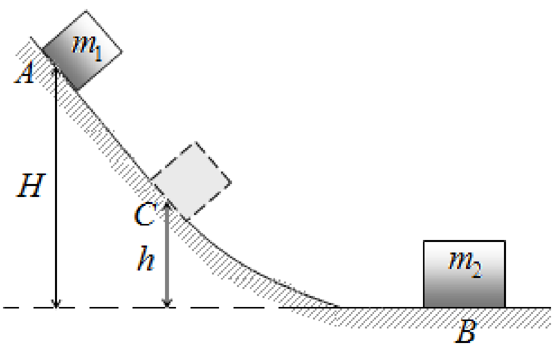
The projections on the $\vec{v}_1$-directions are

$$\begin{cases} u_1 = \dfrac{\mp 2m_2 v_2 + (m_1 - m_2)v_1}{m_1 + m_2}, \\ u_2 = \dfrac{2m_1 v_1 \mp (m_2 - m_1)v_2}{m_1 + m_2}, \end{cases}$$

As a result, considering that $v_2 = 0$, we obtain the speeds of the blocks after collision

$$u_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_1 = \left(\frac{3-6}{3+6}\right)\cdot 14 = -4.67 \text{ m/s},$$

$$u_2 = \left(\frac{2m_1}{m_1 + m_2}\right)v_1 = \left(\frac{2\cdot 3}{3+6}\right)\cdot 14 = 9.33 \text{ m/s}.$$



b) The speed u_1 of the first block is negative, so this block after collision moves to the left (in the opposite direction), and it will head back up the slope. Now we can use again the energy conservation to find the height h. For the trip back up the slope, the initial energy is

$$W_k = \frac{m_1 u_1^2}{2}.$$

When the block reaches maximum height h (point C), its speed is zero, so its energy is potential energy $W_p = m_1 g h$.

Conservation of energy gives $W_k = W_p$, or

$$\frac{m_1 u_1^2}{2} = m_1 g h,$$

Mass m_1 will travel back up the slope to a height

$$h = \frac{u_1^2}{2g} = \frac{4.67^2}{2 \cdot 9.8} = 1.1 \text{ m}.$$

Problem 2.59

A 2 *kg bouncy ball is dropped from a height of* 10 *meters, hits the floor and returns to its original height. What was the change in momentum of the ball upon impact with the floor? What was the impulse provided by the floor?*

Solution

To find the change in momentum of the ball we must find the velocity of the ball just before it hits the ground. To do so, we must rely on the conservation of mechanical energy. The ball was dropped from a height h= 10 meters, and so it had a potential energy of mgh. This energy is converted completely to kinetic energy by the time the ball hits the floor, therefore, $W_p = W_k$ and

$$\frac{mv^2}{2} = mgh,$$

Thus the ball hits the ground with a velocity

$$v = \sqrt{2gh} = \sqrt{2 \cdot 9.8 \cdot 10} = 14 \text{ m/s}.$$

The same argument can be made to find the speed with which the ball bounced back up. When the ball is at ground level, all of the energy of the system is kinetic energy. As the ball bounces back up, this energy gets converted to gravitational potential energy. If the ball reaches the same height it was dropped from, then, we can deduce that the ball leaves the ground with the same speed with which it hit the ground, though in a opposite direction. Thus the change in momentum, is

$$\Delta p = p_2 - p_1 = mv_2 - mv_1 = 2(14) - 2(-14) = 56 \text{ kg·m/s}.$$

The ball's momentum changes by 56 kg·m/s.

We are next asked to find the impulse provided by the floor. By the impulse–momentum theorem, a given impulse causes a change in momentum. Since we have calculated the change in momentum, we already know the impulse. It is equal to 56 kg·m/s.

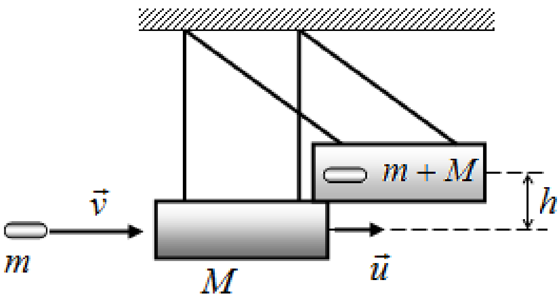
Problem 2.60

A bullet of mass m = 3 *g, travelling at a velocity of* v = 500 *m/s, imbeds itself in the wooden block of a ballistic pendulum. If the wooden block has a mass of* M = 0.3 *kg, to what height does the "bullet-block" combination rise?*

Solution

Since the bullet and the block "stick together" after the collision, this is a perfectly inelastic collision. Momentum is conserved but kinetic energy is not.

The law of conservation of linear momentum gives



$$mv + Mv_{block} = (m + M)u.$$

Since $v_{block} = 0$,

$$mv = (m + M)u,$$

$$u = \frac{mv}{m + M}.$$

After the collision the energy is conserved as long as non-conservative forces do not act on the system, hence,

$$W_k = W_p,$$

$$\frac{(m + M)u^2}{2} = (m + M)gh.$$

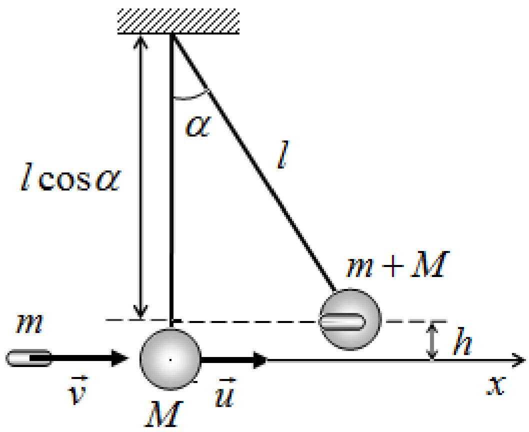
As a result, the “bullet-block” combination rises to the height

$$h=\frac{u^2}{2g}=\frac{(mv)^2}{2g(m+M)^2}=\frac{\left(3\cdot 10^{-3}\cdot 500\right)^2}{2\cdot 9.8\cdot\left(3\cdot 10^{-3}+0.3\right)^2}=1.25\ \text{m}.$$

Problem 2.61

The bullet moving horizontally hits the sphere suspended by weightless rigid rod. The bullet mass m is 1000 *times less than the sphere mass M. The distance between the sphere centre and the pivot point is* l = 1 *m. Find the velocity of the bullet if the rod makes an angle* $\alpha = 10°$ *with vertical.*

Solution



We’ll use the laws of conservation of energy and of linear momentum for the closed system “bullet-sphere” taking into account that the interaction between the bullet and the sphere is the example of inelastic collision of two objects. Its result is the motion of two interacted objects as a whole. It follows that

$$m\vec{v}+M\vec{V}=(m+M)\vec{u},$$

where $\vec{v}$ and $\vec{V}$ are the velocities of the bullet and the sphere before collision, and $\vec{u}$ is the velocity of the sphere with the bullet inside it after the collision.

As the sphere was at rest before collision ($V=0$), the projection of this equation on the horizontal x-axis is

$$mv=(m+M)u,$$

$$u=\frac{mv}{m+M}.$$

The kinetic energy of the moving “bullet-sphere” combination is transferred to its potential energy. Using the law of conservation of energy we obtain

$$\frac{(m+M)u^2}{2}=(m+M)gh.$$

As $h = l - l\cos\alpha = l(1-\cos\alpha)$, then $u = \sqrt{2gl(1-\cos\alpha)}$ and

$$v = \frac{(m+M)u}{m} = \frac{(m+M)\sqrt{2gl(1-\cos\alpha)}}{m}.$$

Taking into account that $M = 1000m$ we obtain

$$v = 1001\sqrt{2gl(1-\cos\alpha)} = 1001\sqrt{2\cdot 9.8\cdot 1(1-\cos 10^\circ)} = 546 \text{ m/s}.$$

Problem 2.62

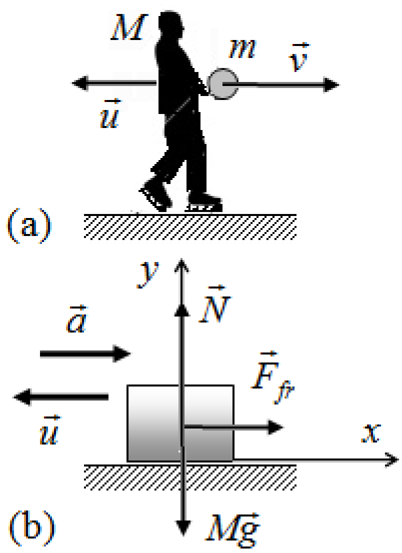
The skater of mass M = 70 kg standing on the ice throws a stone of m = 3 kg in a horizontal direction with a speed v = 8 m/s. Find the distance of recoil if the coefficient of friction between the ice and the skates $\mu = 0.02$.

Solution

This problem may be solved by two methods based on dynamics and kinematics, and using the conservation laws. Let's consider the two.

a) The net linear momentum of the isolated (closed) "skater–stone" system is conserved, hence,

$$(M+m)\vec{v}_0 = M\vec{u} + m\vec{v},$$



Firstly, the skater and the stone were at rest ($v_0 = 0$). Consequently, the linear momentum of the system was equal to zero. Throwing the stone, the skater began to move in the direction opposite to the direction of stone motion, so the equation in projections on x-axis is

$$0 = -Mu + mv.$$

This implies that the skater's speed is

$$u = \frac{mv}{M}.$$

According to the Newton's 2nd Law the equation of the skater motion is

$$M\vec{a} = M\vec{g} + \vec{N} + \vec{F}_{fr},$$

and its x and y projections are

$$\begin{cases} Ma = F_{fr}, \\ 0 = N - Mg. \end{cases}$$

Allow for the fact that friction force $F_{fr} = \mu N$ and normal force is $N = Mg$, we can write that $Ma = \mu N = \mu Mg$ and the magnitude of acceleration of the skater is

$$a = \mu g .$$

The decelerated skater motion before his stop may be described by kinematic equations

$$\begin{cases} v = u - at, \\ s = ut - \dfrac{at^2}{2}. \end{cases}$$

Since $v = 0$ and $u = at$, the travelled distance is

$$s = \frac{u^2}{2a} = \frac{u^2}{2\mu g} = \frac{m^2 v^2}{2\mu M^2 g} = \frac{9 \cdot 64}{2 \cdot 0.02 \cdot 4900 \cdot 9.8} = 0.3 .$$

b) The skater speed using the law of conservation of linear momentum is $u = \dfrac{mv}{M}$ (see above).

According to the law of conservation of energy, the skater expends the kinetic energy in the work against the friction force

$$A_{fr} = \Delta W_k ,$$

$$A_{fr} = \left(\vec{F}_{fr}, \vec{s} \right) = F_{fr} \cdot s \cdot \cos\alpha = -F_{fr} \cdot s ,$$

where $\cos\alpha = -1$, as the friction force is directed opposite to the skater displacement $\vec{s}$.

The increment of kinetic energy is

$$\Delta W_k = 0 - \frac{Mu^2}{2} = -\frac{Mu^2}{2} .$$

Then

$$-F_{fr} \cdot s = -\frac{Mu^2}{2}.$$

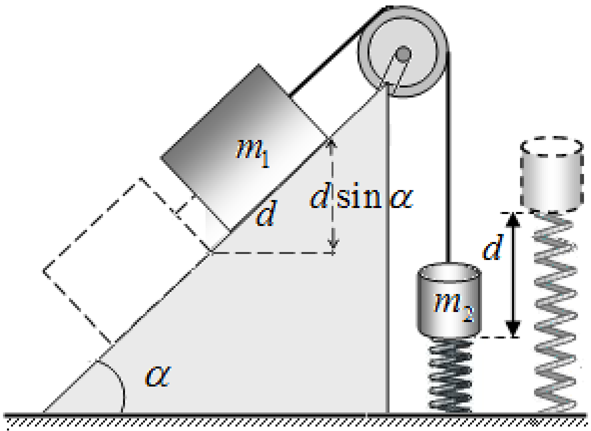
The distance until the stop is

$$s = \frac{Mu^2}{2F_{fr}} = \frac{Mu^2}{2\mu \cdot Mg} = \frac{u^2}{2\mu g} = \frac{m^2v^2}{2\mu \cdot M^2 g} = \frac{9 \cdot 64}{2 \cdot 0.02 \cdot 4900 \cdot 9.8} = 0.3.$$

We obtained the same result as by the first method.

Problem 2.63

A block of mass m_1 = 10 *kg on frictionless incline is connected to a block of mass* m_2 = 30 *kg by a massless string that passes over a light, frictionless pulley. The* 30-*kg block is connected to the unstretched spring that has negligible mass and a force constant of* k = 250 *N/m. The* 10-*kg block is pulled a distance* d = 15 *cm down the incline of angle* $\alpha = 45°$ *and released from rest. Find the speed of each block when the spring is again unstretched.*



Solution

The problem may be solved using the law of conservation of energy because the system is isolated. Both masses are initially at rest, so their kinetic energies are zero. Zero potential energy is defined as the final heights of the masses, i. e., the load m_1 is in its top position, and the load m_2 is in position determined by unstretched spring. As a result of the motion the increments of the potential energies of the first and the second blocks, respectively, are I

$$\Delta W'_p = W'_{p2} - W'_{p1} = m_1 g \cdot (-d) \cdot \sin\alpha - 0 = -m_1 g \cdot d \cdot \sin\alpha.$$

$$\Delta W''_p = W''_{p2} - W''_{p1} = m_2 g \cdot d - 0 = m_2 g \cdot d.$$

The increment of the potential energy of the spring is

$$\Delta W_p''' = W_{p2}''' - W_{p1}''' = \frac{kd^2}{2} - 0 = \frac{kd^2}{2}.$$

The total increment of the potential energy of the system is

$$\Delta W_p = \Delta W_p' + \Delta W_p'' + \Delta W_p''' = -m_1 g \cdot d \cdot \sin\alpha + m_2 g \cdot d + \frac{kd^2}{2}.$$

The increments of the kinetic energy of the first and second blocks, respectively, are

$$\Delta W_k{}' = W_{k2}' - W_{k1}' = \frac{mv_1^2}{2} - 0 = \frac{m_1 v_1^2}{2},$$

$$\Delta W_k{}'' = W_{k2}'' - W_{k1}'' = \frac{m_2 v_2^2}{2} - 0 = \frac{m_2 v_2^2}{2}$$

Both blocks move at same speed because they are connected by the unstretched cord, $v_1 = v_2 = v$. The increment of the total kinetic energy of the system is

$$\Delta W_k = \Delta W_k{}' + \Delta W_k{}'' = \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} = \frac{m_1 v^2}{2} + \frac{m_2 v^2}{2}$$

This increment of the total potential energy of the system is equal to the increment of the kinetic energy, $\Delta W_p = \Delta W_k$.

$$-m_1 g \cdot d \cdot \sin\alpha + m_2 g \cdot d + \frac{kd^2}{2} = \frac{m_1 v^2}{2} + \frac{m_2 v^2}{2}.$$

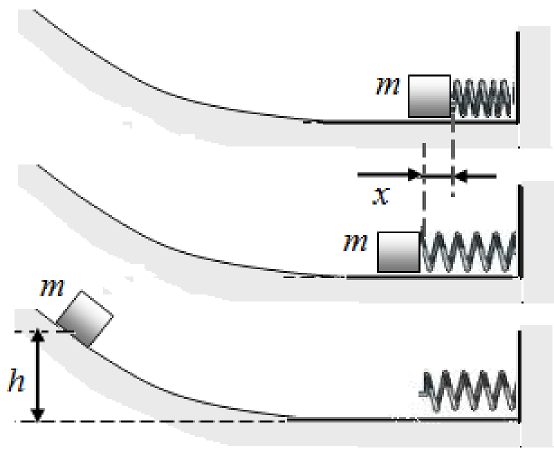
$$v = \sqrt{\frac{2\left(-m_1 g \cdot d \cdot \sin\alpha + m_2 g \cdot d\right) + k \cdot d^2}{m_1 + m_2}} =$$

$$= \sqrt{\frac{2\left(-10 \cdot 9.8 \cdot 0.15 \cdot 0.707 + 30 \cdot 9.8 \cdot 0.15\right) + 250 \cdot 0.15^2}{10 + 30}} = 1.35 \text{ m/s}.$$

Problem 2.64

In the diagram below, the spring has a force constant of 5000 *N/m, the block has a mass of* 5 *kg, and the height h of the hill is* 8 *m. Determine the compression of the spring such that the block just makes it to the top of the hill. Assume that there are no non-conservative forces involved, and g* =10 *m/s*^{2}.

Solution



Work-Energy Theorem may be used for the solution of this problem. The block obtained its energy due to compressed spring W_{ps}. This energy is transformed into the kinetic energy W_k of the block, which, in its turn, is transformed into its potential energy W_p.

$$W_{ps} = W_k = W_p.$$

Consequently, $W_{ps} = W_p$, and

$$\frac{kx^2}{2} = mgh.$$

Finally,

$$x = \sqrt{\frac{2mgh}{k}} = \sqrt{\frac{2 \cdot 5 \cdot 10 \cdot 8}{5000}} = 0.4 \text{ m}.$$

Problem 2.65

A 10 *kg block is released from point A at the height of* 3 *m. The track is frictionless except for the portion BC, of length* 6 *m. The block travels down the track, hits a spring of force constant k* = 2250 *N/m, and compresses it* 0.3 *m from its equilibrium position before coming to rest momentarily. Determine the coefficient of kinetic friction between surface BC and block.*

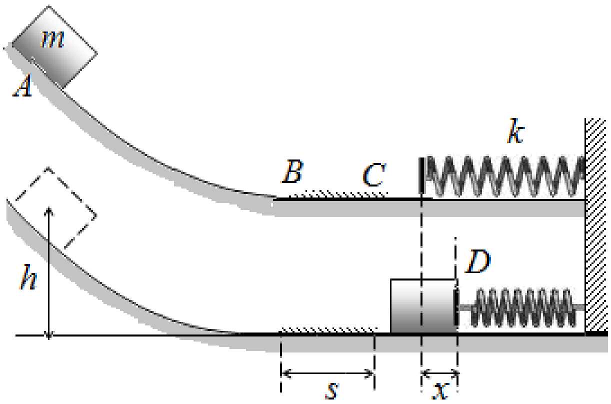
Solution

The forces which act on the mass as it descends and goes on to squish the spring are: gravity, the spring force and the force of kinetic friction as it slides over the rough part BC. Gravity and the spring force are conservative forces, so we will keep track of them with the potential energy associated with these forces. Friction is a non-conservative force, but in this case we can calculate the work that it does. Then, we can use the energy conservation principle,

$$\Delta W_k + \Delta W_p = A_{fr}, \quad (1)$$

to find the unknown quantity, namely μ for the rough surface BC.

The block released at point A so its initial speed (hence, kinetic energy) is zero, $W_{k1} = 0$. Initial potential energy is



$$W_{p1} = mgh.$$

The final position of the block is when it has maximally compressed the spring. At this final point D, the mass is again at rest, so its kinetic energy is zero: $W_{k2} = 0$. Being at zero height, it has no gravitational potential energy ($W_{p2} = 0$ but now since there is a compressed spring, there is stored (potential) energy in the spring W_{ps}. This energy is given by

$$W_{ps} = \frac{kx^2}{2}.$$

This energy is final potential energy of the system.

The total mechanical energy of the system changes because there is a non-conservative force (friction F_{fr}) which does work A_{fr}. As the mass slides over the rough part BC, the vertical forces are gravity $m\vec{g}$ (downward) and the upward normal force of the surface $\vec{N}$. As there is no vertical motion, $N = mg$. The magnitude of the force of kinetic friction is

$$F_{fr} = \mu N = \mu mg.$$

Now we have everything we need to substitute into the energy balance condition (1), i. e.,

$$\Delta W_k = W_{k2} - W_{k1} = 0,$$

$$\Delta W_p = W_{ps} - W_{p1} = \frac{kx^2}{2} - mgh,$$

$$A_{fr} = \left(\vec{F}_{fr}, \vec{s}\right) = F_{fr} \cdot s \cdot \cos\alpha = \mu \cdot mg \cdot s \cos 180^0 = -\mu \cdot mg \cdot s.$$

Substitution gives

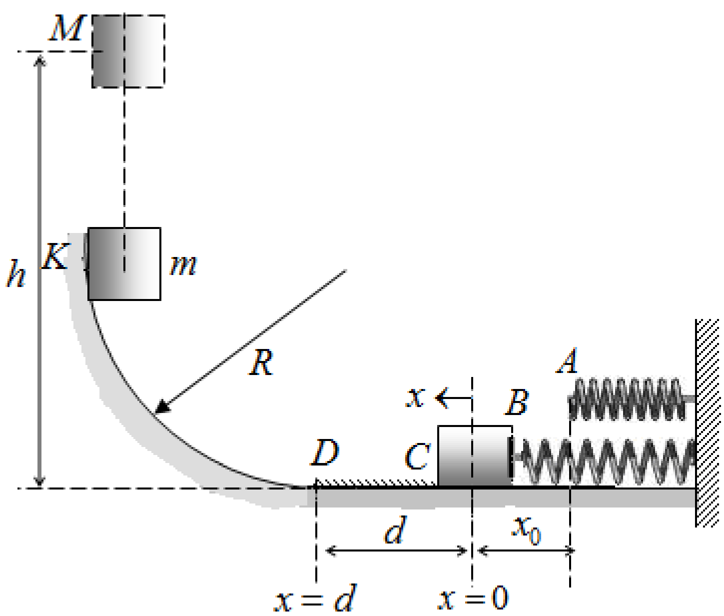
$$\frac{kx^2}{2} - mgh = -\mu \cdot mg \cdot s,$$

$$\mu = \frac{kx^2 - 2mg \cdot h}{-2mg \cdot s} = \frac{2250 \cdot 0.09 - 2 \cdot 10 \cdot 9.8 \cdot 3}{-2 \cdot 10 \cdot 9.8 \cdot 6} = \frac{202.5 - 588}{-1176} = 0.328.$$

The coefficient of kinetic friction for the rough surface is 0.328.

Problem 2.66

An object of mass m is released from an initial state of rest from a spring of constant k that has been compressed a distance x_0. After leaving the spring (at the position $x = 0$ when the spring is unstretched) the object travels a distance d along a horizontal track that has a coefficient of friction that varies with position as $\mu = \mu_0 + \mu_1\left(\frac{x}{d}\right)$. Following the horizontal track, the object enters a quarter turn of a frictionless loop whose radius is R. Finally, after exiting the quarter turn of the loop the object travels vertically upward to a maximum height, h, (as measured from the horizontal surface). Find the maximum height, h, that the object attains.



Solution

The initial state is when the spring is compressed the distance x_0 (point A) and the final state is when the object is at its maximum height (point M). The initial energy is potential energy of compressed spring $W_{p1} = \frac{kx_0^2}{2}$, and the final energy is potential energy of the object at the height h, $W_{p2} = mgh$.

The work A_{fr} is done by nonconservative friction force F_{fr} . The magnitude of the friction force is $F_{fr} = \mu mg$ and the work is

$$A_{fr} = -\int F_{fr} \cdot dx = -mg\int \mu \cdot dx .$$

$$A_{fr} = -mg\int_0^d \left(\mu_0 + \mu_1 \left(\frac{x}{d} \right) \right) dx = -mg\left(\mu_0 d + \mu_1 \frac{d}{2} \right) = -mgd\left(\mu_0 + \frac{\mu_1}{2} \right).$$

Note that in the figure to the problem, x increases from right to left. In any event, the work done by friction is negative.

The law of conservation of energy gives

$$mgh - \frac{kx_0^2}{2} = -mg \cdot d\left(\mu_0 + \frac{\mu_1}{2} \right),$$

which is easily solved for

$$h = \frac{kx_0^2}{2mg} - d\left(\mu_0 + \frac{\mu_1}{2} \right).$$

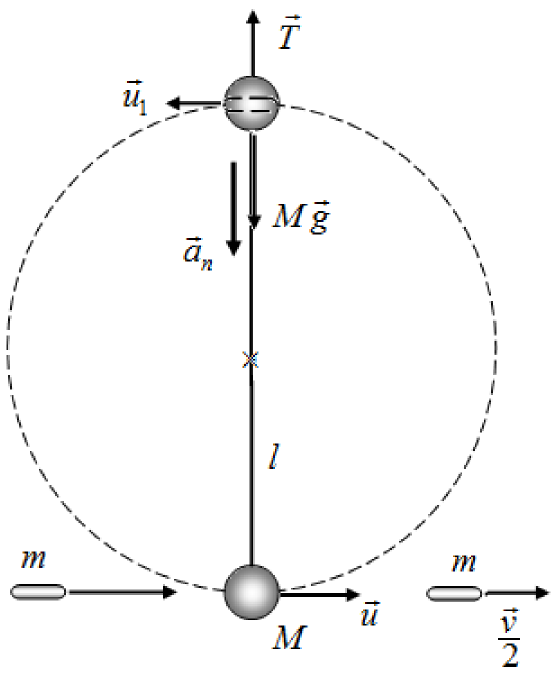
The analysis of the result shows that if that result were negative, the friction would be enough to stop the object before it entered the loop.

Problem 2.67

A bullet of mass m and speed v passes completely through a pendulum bob of mass M. The bullet emerges with a speed $v/2$. What is the minimum value of v such that the pendulum bob will barely swing through a complete vertical circle? Solve this problem if the pendulum bob is suspended by a) a stiff rod of length l and negligible mass, and b) a weightless inextensible rope.

Solution

a) We have following sequence of events: the bullet has a very rapid and strong interaction with the pendulum bob, where is quickly passes through,



imparting a velocity to the bob which at first will have a horizontal motion; and the bob swings upward and, as we are told, will get up to the top of the vertical circle.

We show the collision in Figure. In this rapid interaction there are no net external forces acting on the system that we need to worry about. So its total momentum will be conserved. The total horizontal momentum before the collision is equal to the momentum of moving bullet $\vec{p} = m\vec{v}$. If after the collision the bob has velocity $\vec{u}$, then the total momentum of the system that consists of the bullet with momentum $\vec{p}_1' = m\frac{\vec{v}}{2}$ and the pendulum bob which momentum is $\vec{p}_2' = M\vec{u}$. As a result, the total linear momentum of the system after collision is $\vec{p}' = \vec{p}_1' + \vec{p}_2' = m\frac{\vec{v}}{2} + M\vec{u}$.

Conservation of momentum, $\vec{p} = \vec{p}'$, gives

$$m\vec{v} = m\frac{\vec{v}}{2} + M\vec{u},$$

or, since all velocities directed horizontally,

$$mv = m\frac{v}{2} + Mu.$$

The speed of pendulum bob is equal to

$$u = \frac{mv}{2M}. \tag{1}$$

Now consider the trip of the pendulum bob up to the top of the circle (it must get to the top, by assumption). There are no friction-type forces acting on the system as bob moves, so mechanical energy is conserved.

If we measure height from the bottom of the swing, then the initial potential energy is zero while the initial kinetic energy is

$$W_k = \frac{Mu^2}{2}.$$

Now suppose at the top of the swing mass M has speed u_1. Its height is $2l$ and its potential energy is $W_p = Mg \cdot 2l$ so that its final energy is

$$W_2 = \frac{Mu_1^2}{2} + 2Mgl.$$

So that conservation of energy gives

$$W_k = W_2,$$

$$\frac{Mu^2}{2} = \frac{Mu_1^2}{2} + 2Mgl. \quad (2)$$

In the top point of trajectory the bob is moving on a circular path with (instantaneous) speed u_1, and with normal (centripetal) acceleration $a_n = \frac{u_1^2}{l}$, hence, according to the Newton's 2nd Law

$$M\vec{a}_n = \vec{F},$$

where $\vec{F}$ is the net force.

A drawing of the forces acting on bob of mass M at the top of the swing is shown in Figure. Gravity pulls down with a force $M\vec{g}$. The bob was suspended by a stiff rod and such an object can exert a force (the tension $\vec{T}$) inward or outward along its length, whereas a string can only pull inward.

$$M\vec{a}_n = M\vec{g} + \vec{T},$$

or

$$M\frac{u_1^2}{l} = Mg \pm T. \quad (3)$$

Since $\vec{T}$ can be positive or negative, u_1 can take on any value, even it could be zero. What condition are we looking for which corresponds to the smallest value of the bullet speed v? We note that as v gets bigger, so does u (the bob's initial speed). As u increases, so does u_1, as we see from conservation of energy (2). But it is entirely possible for u_1 to be zero, and that will give the smallest possible value of v. That would correspond to the case where the bob picked up enough speed to just barely make it to the top of the swing. And when the bob goes past the top point then gravity moves it along through the full swing.

So with $u_1 = 0$, equation (2) according to the conservation of energies law gives

$$\frac{Mu^2}{2} = 2Mgl,$$

and the speed of the bob after its interaction with the bullet is

$$u = \sqrt{4gl}.$$

Putting this result into equation (1), we have

$$\sqrt{4gl} = \frac{mv}{2M}.$$

Finally, solve for v, we obtain the minimum speed of the bullet allowing the bob to rich the top point of trajectory

$$v = \frac{2M}{m}\sqrt{4gl} = \frac{4M}{m}\sqrt{gl}.$$

b) When the bob is suspended by the rope it is necessary to have tight rope all the time except the top point of trajectory.

Then equation (3) when $T = 0$, is

$$M\frac{u_1^2}{l} = Mg,$$

$$u_1 = \sqrt{gl}.$$

Equation (2) according to the conservation of energies law gives us

$$\frac{Mu^2}{2} = \frac{Mgl}{2} + 2Mgl = 2.5Mgl,$$

and

$$u = \sqrt{5gl}\ .$$

Putting this result into the equation (1), we obtain

$$\sqrt{5gl} = \frac{mv}{2M}.$$

Finally, solve for v and obtain the speed of the bullet

$$v = \frac{2M}{m}\sqrt{5gl}\ .$$

Problem 2.68

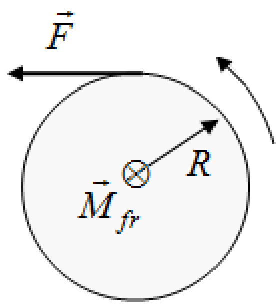
The tangential force F =100 N is applied to the rim of the homogeneous disc of R = 0.2 m. At the rotation the frictional torque M_{fr} =5 N·m acts on the disc. Find the disc mass if its acceleration is $\varepsilon = 100$ rad/s^2.

Solution

The disc motion is described by the Newton's 2nd Law for rotation

$$I\vec{\varepsilon} = \vec{M}\ ,$$

where I is a moment of inertia of the disc, ε is its angular acceleration, M is the net torque of the forces applied to the disc.



Moment of inertia of disc about the axis passing through its center of mass is

$$I = \frac{mR^2}{2}.$$

Since the vectors of the angular acceleration and driving torque are of the same direction along the axis of rotation, and the frictional torque is of opposite direction, the net torque is

$$M = M_F - M_{fr} = F \cdot R - M_{fr}\ .$$

Newton's 2nd Law for rotation takes on form

$$\frac{mR^2\varepsilon}{2} = F \cdot R - M_{fr}\ .$$

Solving for the disc mass gives

$$m = \frac{2(F \cdot R - M_{fr})}{R^2 \varepsilon} = \frac{2\left(100 \cdot 0.2 - 5\right)}{0.04 \cdot 100} = 7.5 \text{ kg.}$$

Problem 2.69

If a wheel has moment of inertia of 5 *kg·m², a) what angular speed does the wheel attain if* 10^5 *Joule of work is done in producing rotational kinetic energy? b) What torque is required to bring the wheel to rest in* 25 *seconds?*

Solution

a) The work was done for changing the rotational kinetic energy from rest, $W_{k1} = 0$, to $W_{k2} = \frac{I\omega^2}{2}$, consequently,

$$A = \Delta W_k = W_{k2} - W_{k1} = W_{k2} = \frac{I\omega^2}{2},$$

where I is the moment of inertia of the wheel, and ω is its initial angular speed which is equal to

$$\omega = \sqrt{\frac{2A}{I}} = \sqrt{\frac{2 \cdot 10^5}{5}} = 200 \text{ rad/s.}$$

b) The decelerated motion of the disc is described by kinematic equation

$$\omega_1 = \omega - \varepsilon t.$$

where $\omega_1 = 0$.

Hence,

$$\varepsilon = \frac{\omega}{t}.$$

The braking torque is

$$M = I\varepsilon = \frac{I\omega}{t} = \frac{5 \cdot 200}{25} = 40 \text{ N·m.}$$

Problem 2.70

A rope is wrapped around a solid cylindrical drum. The drum has a fixed frictionless axle. The mass of the drum is 100 *kg and it has a radius of* R = 50 *cm. The other end of the rope is tied to a block,* M = 10 *kg. What is the angular acceleration of the drum? What is the linear acceleration of the block? What is the tension in the rope? Assume that the rope does not slip.*

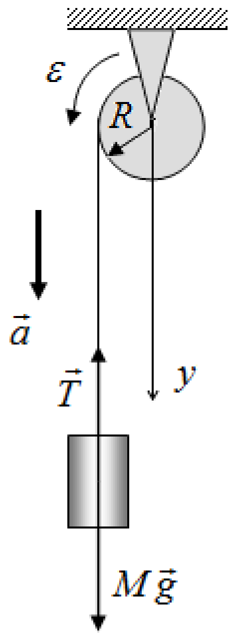
Solution

The forces acting on the block are gravity $M\vec{g}$ and tension $\vec{T}$. Tension $\vec{T}$ acting on the drum creates a torque. Note that rope and, therefore, tension are tangential to the drum and thus normal to the radius. Two other forces acting on the drum (the normal force from the axle and the gravity) pass through its centre of mass and thus do not create torque. The drum accelerates counter-clockwise as the block moves down.

Since the rope is wrapped around the drum and moves along with the drum, the linear acceleration of the rope related to the angular acceleration of the drum (and the part of the rope which is bedded on the drum) as $a = \varepsilon R$.

The load moves along straight line and the drum rotates, therefore, their motions have to be described by Newton's 2nd law for the translational and rotational motions, respectively.

$$\begin{cases} M\vec{a} = M\vec{g} + \vec{T}, \\ I\vec{\varepsilon} = \vec{M}. \end{cases}$$



The projection of the first equation on y-axis is $Ma = Mg - T$.

The second equation changes to $\frac{mR^2}{2} \cdot \frac{a}{R} = T \cdot R$ after the substitution of the moment of inertia of cylinder $I = \frac{mR^2}{2}$, the angular acceleration $\varepsilon = \frac{a}{R}$ and the torque $M = TR$. As a result, it simplifies to $T = \frac{ma}{2}$.

Substitution of the tension into the first equation yields

$$Ma = Mg - \frac{ma}{2},$$

$$a = \frac{Mg}{M + 0.5m} = \frac{10 \cdot 9.8}{10 + 50} = 1.63 \ \text{m/s}^2.$$

The angular acceleration of the drum is

$$\varepsilon = \frac{a}{R} = \frac{1.63}{0.5} = 3.27 \ \text{rad/s}^2,$$

and the tension is

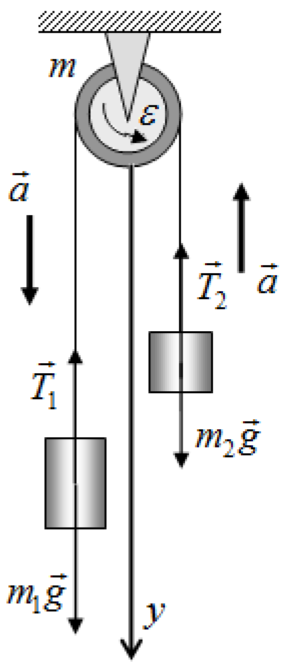
$$T = \frac{ma}{2} = \frac{10 \cdot 1.63}{2} = 81.5 \ \text{N}.$$

Problem 2.71

Two blocks (m_1 = 2 kg and m_2 = 1 kg) are connected by rope over a frictionless pulley as shown in the Figure. The pulley is a cylinder of mass m =1 kg. What is the acceleration of the blocks and the tension in the rope on either side of the pulley?

Solution

In this problem we must use both the linear and rotational versions of Newton's 2nd Law, because there are two objects that take part in translational motion, and one object rotates. We have to write down three equations. Remember that the tension must be different on either side of the pulley or the pulley would not rotate.



$$\begin{cases} m_1\vec{a} = m_1\vec{g} + \vec{T}_1, \\ m_2\vec{a} = m_2\vec{g} + \vec{T}_2, \\ I\vec{\varepsilon} = \vec{M}. \end{cases}$$

Taking into account that the moment of inertia of solid disc is $I = \dfrac{mR^2}{2}$ and the linear acceleration of the rope and the angular acceleration of the disc relate as $a = \varepsilon R$, we can write

$$\begin{cases} m_1 a = m_1 g - T_1, \\ -m_2 a = m_2 g - T_2, \\ \dfrac{m\cancel{R}^{\cancel{2}}}{2} \cdot \dfrac{a}{\cancel{R}} = \left(T_1 - T_2\right)\cancel{R}. \end{cases}$$

or

$$\begin{cases} m_1 a = m_1 g - T_1, \\ -m_2 a = m_2 g - T_2, \\ \dfrac{ma}{2} = T_1 - T_2. \end{cases}$$

Solution of this system gives

$$a = \frac{\left(m_1 - m_2\right) g}{m_1 + m_2 + m/2} = \frac{(2-1)\cdot 9.8}{2+1+0.5} = 2.8 \text{ m/s}^2,$$

$$T_1 = \frac{m_1 g\left(2m_2 + m/2\right)}{m_1 + m_2 + m/2} = 14 \text{ N},$$

$$T_2 = \frac{m_2 g\left(2m_1 + m/2\right)}{m_1 + m_2 + m/2} = 12.6 \text{ N}.$$

Problem 2.72

Cylinder of mass m = 20 kg and radius R = 0.5 m rotates about the axis passing through its center according to the dependence $\varphi(t) = A + Bt^2 - Ct^3$, *where B = 2 rad/s², C = 2 rad/s³. Find the time dependence for torque change, and the magnitude of the torque at the instant of time t = 5 s.*

Solution

According to the Newton's 2nd Law for rotation, the torque depends on the moment of inertia and the angular acceleration of cylinder as

$$\vec{M} = I\vec{\varepsilon}\,,$$

where $I = \dfrac{mR^2}{2}$ is the moment of inertia of the cylinder.

Time dependencies of angular speed and angular acceleration can be determined by taking the derivatives with respect with time of the given dependence $\varphi(t)$.

$$\omega(t)=\frac{d\varphi}{dt}=2Bt-3Ct^2,$$

$$\varepsilon(t)=\frac{d\omega}{dt}=2B-6Ct.$$

Then

$$M(t)=I\varepsilon(t)=\frac{mR^2}{2}(2B-6Ct).$$

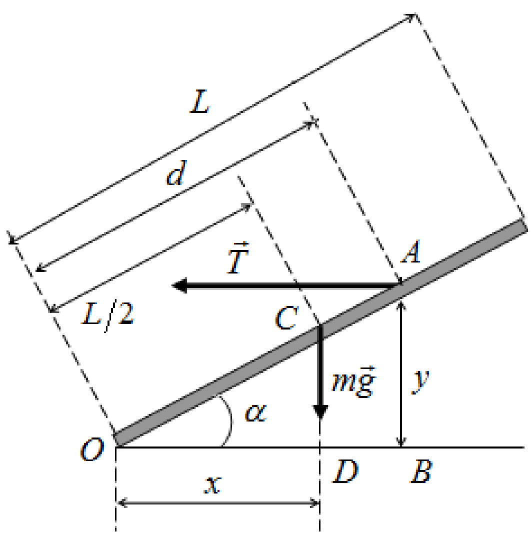
$$M=\frac{20\cdot 0.25}{2}(2\cdot 2-6\cdot 2t)=10-30t\big|_{t=5}=140\ \text{N}\cdot\text{m}.$$

Problem 2.73

A trapdoor on a stage has a mass of 22 *kg and a width of* 1.6 *m (hinge side to handle side). The door can be treated as having uniform thickness and density. A small handle on the door is* 1.5 *m away from the hinge side. A rope is tied to the handle and used to raise the door. At one instant, the rope is horizontal, and the trapdoor has been partly opened so that the handle is* 1.1 *m above the floor. What is the tension* T *in the rope at this time?*

Solution

The rope tied to the handle of the trap door makes it rotate about the axis passing along the edge with hinges (point O). Two forces act on the door: the gravity $m\vec{g}$ applied at the center of mass C located at the distance $L/2=1.6/2=0.8$ m from O (since the door has uniform density and thickness), and the tension T. The tension of the rope exerts counter-clockwise torque M_1 about the hinge which balances the weight of the trap door which is exerting a clockwise torque M_2.



Consider the triangle ΔABO: $OA = 1.5$ m and $AB = y = 1.1$ m, the angle

$$\alpha = \arcsin(AB/AO) = \arcsin(1.1/1.5) = \arcsin 0.73 = 47.2°.$$

The arm of the gravity force is the distance

$$x = OD = \frac{L}{2} \cdot \cos\alpha = 0.8 \cdot \cos 47.2° = 0.54 \text{ m}.$$

A trapdoor is in the state of rotational equilibrium, therefore, the vector sum of the torques is equal to zero. In other words, net torque that rotates clockwise has to be equal to net torque that rotates counter-clockwise.

Thus, $M_2 - M_1 = 0$,

$$mg \cdot x = T \cdot y,$$

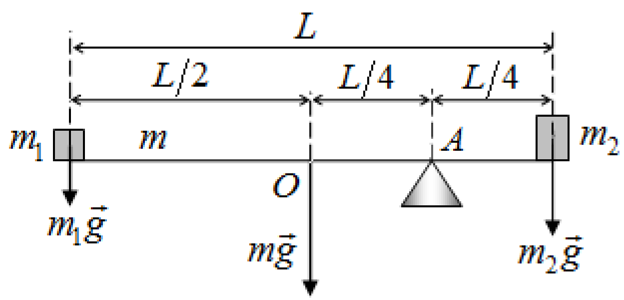
$$T = \frac{mgx}{y} = \frac{22 \cdot 9.8 \cdot 0.54}{1.1} = 105.8 \text{ N}.$$

Problem 2.74

The system shown in the figure is held initially at rest. Calculate the angular acceleration of the system as soon as it is released. You can treat m_1 =2 kg and m_2 =10 kg as point masses located on either end of the rod of mass m =24 kg and length L =12 m.

Solution

The moment of inertia of the rod about the axis passing through its center of mass is $I_0 = \dfrac{mL^2}{12}$.



According to the parallel axis (Huygens-Steiner) theorem the moment of inertia about the axis passing through the point A is

$$I_r = I_0 + mx^2 = \frac{mL^2}{12} + m\left(\frac{L}{4}\right)^2 = \frac{7mL^2}{48} = \frac{7 \cdot 24 \cdot 12^2}{48} = 504 \text{ kg·m²}.$$

Taking into account that the mass m_1 is separated by distance $3L/4$ from the point A, and the mass m_2 is located at distance $l/4$ from the pivot point, the moments of inertia of these point masses are, respectively,

$$I_1 = m_1\left(\frac{3L}{4}\right)^2 = \frac{9m_1L^2}{16} = \frac{9\cdot 2\cdot 12^2}{16} = 162 \text{ kg·m²},$$

$$I_2 = m_2\left(\frac{L}{4}\right)^2 = \frac{m_2L^2}{16} = \frac{10\cdot 12^2}{16} = 90 \text{ kg·m²}.$$

The moment of inertia of the system (the rod with two point masses) is

$$I = I_r + I_1 + I_2 = 504 + 162 + 90 = 756 \text{ kg·m²}.$$

Assuming that the gravity of the rod is applied at its center of mass, we can determine the torque of the system with respect to the point A (counterclockwise direction is positive)

$$M = M_r + M_1 - M_2 = mg\cdot\frac{L}{4} + m_1g\cdot\frac{3L}{4} - m_2g\cdot\frac{L}{4} =$$
$$= \frac{gL}{4}\left(m + 3m_1 - m_2\right) = \frac{9.8\cdot 12\cdot\left(24 + 3\cdot 2 - 10\right)}{4} = 588 \text{ N}\cdot\text{m}.$$

According to the Newton's 2nd Law for rotation $M = I\varepsilon$, and the angular acceleration of the system is

$$\varepsilon = \frac{M}{I} = \frac{588}{756} = 0.78 \text{ rad/s²}.$$

Problem 2.75

Two thin rectangular sheets (0.2 $m\times 0.4\,m$) *are identical. In the first sheet the axis of rotation lies along the* 0.2-*m side, and in the second it lies along the* 0.4-*m side. The same torque is applied to each sheet. The first sheet, starting from rest, reaches its final angular velocity in* 10 *s. How long does it take for the second sheet, starting from rest, to reach the same angular velocity?*

Solution

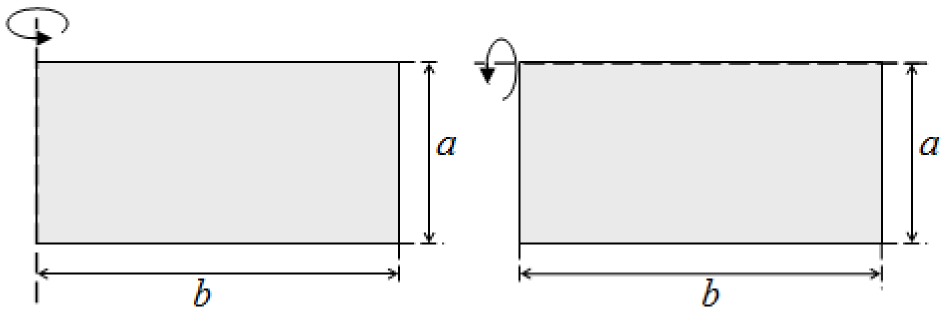
The moment of inertia of the uniform rectangular plate about its side is

$$I = \frac{mL^2}{3},$$

where L is the length of another side (perpendicular to the axis of rotation).

Thus, the moments of inertia for the examined plates are

$$I_1 = \frac{mb^2}{3} \text{ and } I_2 = \frac{ma^2}{3}.$$



Due to the torque acting the plates have the same angular speeds

$$\omega = \varepsilon_1 t_1,$$

$$\omega = \varepsilon_2 t_2.$$

Consequently, the angular accelerations of the plates are given as

$$\varepsilon_1 = \frac{\omega}{t_1} \text{ and } \varepsilon_2 = \frac{\omega}{t_2}.$$

Applying the Newton's 2nd Law for rotation $M = I\varepsilon$ for each plate gives

$$M = I_1 \cdot \varepsilon_1 = \frac{mb^2}{3} \cdot \frac{\omega}{t_1},$$

$$M = I_2 \cdot \varepsilon_2 = \frac{ma^2}{3} \cdot \frac{\omega}{t_2}.$$

Equate expressions for torques

$$\frac{mb^2}{3} \cdot \frac{\omega}{t_1} = \frac{ma^2}{3} \cdot \frac{\omega}{t_2}$$

and obtain the required time

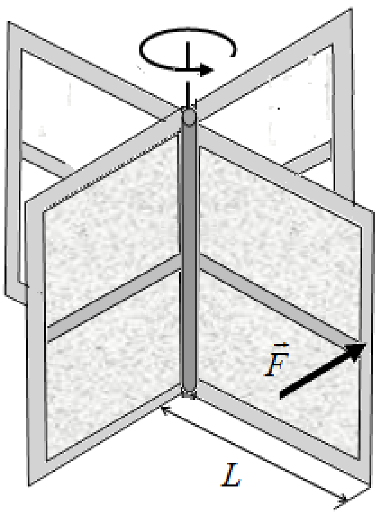
$$t_2 = \left(\frac{a}{b}\right)^2 \cdot t_1 = \left(\frac{0.2}{0.4}\right)^2 \cdot 10 = 2.5\ \text{s}.$$

Problem 2.76

A rotating door is made from four rectangular sections. The mass of each section is m = 80 *kg; the width is* L = 1.3 *m. A person pushes on the outer edge of one section with a force of* F = 60 *N that is directed perpendicular to the section. Determine the magnitude of the door's angular acceleration.*

Solution

The moment of inertia of each rectangular plate about the axis passing through the center of mass is $I_0 = \frac{mL^2}{12}$. The moment of inertia about the parallel axis passing through the edge according the parallel axes theorem is



$$I_1 = I_0 + mx^2 = \frac{mL^2}{12} + m\left(\frac{L}{2}\right)^2 = \frac{mL^2}{3}.$$

The moment of inertia of the door consisting of four sections is

$$I = 4I_1 = \frac{4mL^2}{3} = \frac{4 \cdot 80 \cdot 1.69}{3} = 180.3\ \text{kg·m²}.$$

The torque applied to the door is

$$M = F \cdot L = 60 \cdot 1.3 = 78\ \text{N·m}.$$

Using Newton's 2nd Law for rotation $M = I\varepsilon$ gives the angular acceleration as

$$\varepsilon = \frac{M}{I} = \frac{78}{180.3} = 0.43\ \text{rad/s²}.$$

Problem 2.77

A circular hoop of mass 50 *kg and radius* 50 *cm is rotating at an angular speed of* 120 *revolutions per minute. Calculate its kinetic energy.*

Solution

For kinetic energy calculation we have to know and the angular speed $\omega = 120$ rev/min=2 rev/s and the moment of inertia of the hoop

$$I = mR^2 = 50 \cdot 0.25 = 12.5 \text{ kg·m}^2,$$

Then the kinetic energy of rotating hoop is

$$W_k = \frac{I\omega^2}{2} = \frac{12.5 \cdot 0.04}{2} = 0.25 \text{ J}.$$

Problem 2.78

A sphere of mass 100 *kg and radius* 50 *cm rolls without slipping with the speed of* 5 *cm/s. Calculate its kinetic energy.*

Solution

Since the sphere is moving, it has kinetic energy. The sphere takes part in two motions simultaneously, i. e., the translational motion with kinetic energy $\left(W_k\right)_{trans} = \frac{mv^2}{2}$, and rotational motion with kinetic energy $\left(W_k\right)_{rot} = \frac{I\omega^2}{2}$, where m and $I = \frac{2mR^2}{5}$ are the mass and the moment of inertia of sphere, respectively; v and $\omega = \frac{v}{R}$ are its linear and angular speeds. The total energy is the sum of these two energies:

$$W_k = \frac{mv^2}{2} + \frac{I\omega^2}{2} = \frac{mv^2}{2} + \frac{\cancel{2}m\cancel{R}^2 v^2}{5 \cdot \cancel{2} \cdot \cancel{R}^2} = 0.7mv^2.$$

Problem 2.79

Calculate the kinetic energy of rotation for disc of mass 1 *kg and radius* 0.2 *m rotating at* 30 *rad/min. Find the kinetic energy of this disc if it rolls without slipping, and the points on the disc surface have the angular speed* 30 *rad/min.*

Solution

Kinetic energy of rotating disc may be determined according to the expression

$$W_k = \frac{I\omega^2}{2},$$

where $I = \frac{mR^2}{2} = \frac{1 \cdot 0.04}{2} = 0.02\ \text{kg·m}^2$ is the moment of inertia of the disc, and ω = 30 rad/min = 0.5 rad/s is its angular speed.

Then the kinetic energy of rotation is

$$W_k = \frac{I\omega^2}{2} = \frac{0.02 \cdot 0.25}{2} = 2.5 \cdot 10^{-3}\ \text{J}.$$

When disc rolls without slipping its kinetic energy is sum of kinetic energies of translational and rotational motions. Taking into account that the moment of inertia of disc is $I = \frac{mR^2}{2}$ and liner velocity of the point of the disc surface correlates with angular velocity according to $v = \omega R$, we obtain

$$W_k = \frac{mv^2}{2} + \frac{I\omega^2}{2} = \frac{m\omega^2 R^2}{2} + \frac{mR^2\omega^2}{2 \cdot 2} =$$

$$= 0.75 mR^2\omega^2 = 0.75 \cdot 1 \cdot 0.04 \cdot 0.25 = 7.5 \cdot 10^{-3}\ \text{J}.$$

Problem 2.80

The 25*-kg disc of radius* 0.3 *m is at rest when the constant* 15 *N·m counterclockwise couple is applied. Determine the disc's angular velocity when it has rotated through* 5 *revolutions a) by applying the Newton's 2nd Law for rotation, and b) by applying the principle of work and energy.*

Solution

a) Newton's 2nd Law for rotation $M = I\varepsilon$ gives the angular acceleration of the disc as

$$\varepsilon = \frac{M}{I}.$$

The moment of inertia of the disc is $I = \frac{mR^2}{2}$, therefore, the angular acceleration is

$$\varepsilon = \frac{M}{I} = \frac{2M}{mR^2}.$$

From the kinematics, the equations for rotational accelerated motion from rest are

$$\begin{cases} 2\pi N = \dfrac{\varepsilon t^2}{2}, \\ \omega = \varepsilon t. \end{cases}$$

The first equation gives $t = \sqrt{\frac{4\pi N}{\varepsilon}}$ and angular speed is

$$\omega = \varepsilon t = \sqrt{4\pi N\varepsilon} = \sqrt{4\pi N \cdot \frac{2M}{mR^2}} = \sqrt{\frac{4\pi \cdot M \cdot N}{mR^2}} = \sqrt{\frac{4\pi \cdot 15 \cdot 5}{25 \cdot 0.3^2}} = 20.5 \text{ rad/s}.$$

b) Applying the law of conservation of energy, the work is

$$A = \int_0^{2\pi N} M d\varphi = M \cdot 2\pi N.$$

On the other hand, the work is equal to the change of rotational kinetic energy

$$A = \Delta W_k = W_{k2} - W_{k1} = \frac{I\omega^2}{2} - 0 = \frac{mR^2\omega^2}{4}.$$

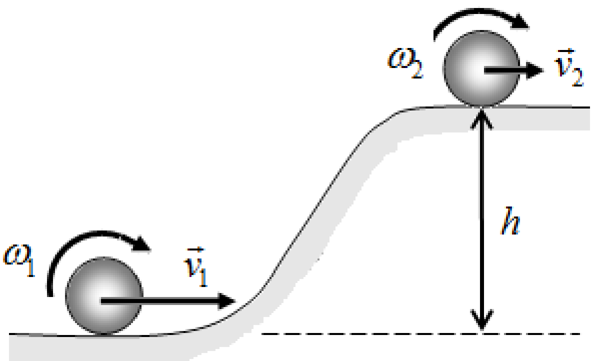
Equating the right hand side of the equations for the work, we obtain

$$M\cdot 2\pi N=\frac{mR^2\omega^2}{4},$$

$$\omega=\sqrt{\frac{4\pi\cdot M\cdot N}{mR^2}}=\sqrt{\frac{4\pi\cdot 15\cdot 5}{25\cdot 0.3^2}}=20.5 \text{ rad/s}.$$

Problem 2.81

A bowling ball encounters a 0.8 *m vertical rise on the way back to the ball rack, as the drawing illustrates. Ignore frictional losses and assume that the mass of the ball is distributed uniformly. The translational speed of the ball is* 4 *m/s at the rise. Find the translational speed at the top.*



Solution

The bowling ball is rolling without slipping, therefore, its energy consists of the kinetic energies of translational (W_{kt}) and rotational (W_{kr}) motions.

$$W_1=W_{p1}+W_{k1}=W_{p1}+W_{kt1}+W_{kr1}=W_{p1}+\frac{mv_1^2}{2}+\frac{I\omega_1^2}{2}.$$

The potential energy is zero at the bottom of the rock. The moment of inertia of the solid sphere of the mass m and radius R is $I=\frac{2mR^2}{5}$, and the angular speed (ω) relates to the linear speed of the ball's center of mass (v) as $v=\omega R$. Then

$$W_{k1}=\frac{mv_1^2}{2}+\frac{\cancel{2}m\cdot \cancel{R}^2\cdot v_1^2}{5\cdot\cancel{2}\cdot\cancel{R}^2}=\frac{7mv_1^2}{10}.$$

At the top of the rack the ball's energy is given by

$$W_2=W_{p2}+W_{k2}=W_{p2}+W_{kt2}+W_{kr2}=mgh+\frac{mv_2^2}{2}+\frac{I\omega_2^2}{2}=mgh+\frac{7mv_2^2}{10}.$$

At the absence of friction, $W_1 = W_2$, therefore,

$$\frac{7mv_1^2}{10} = mgh + \frac{7mv_2^2}{10}.$$

The translational velocity of the ball at the top is

$$v_2 = \sqrt{v_1^2 - \frac{10gh}{7}} = \sqrt{4^2 - \frac{10 \cdot 9.8 \cdot 0.8}{7}} = 2.2 \text{ m/s}.$$

Problem 2.82

The 100-*kg homogenous cylindrical disk of radius* R =0.3 *m is at rest when the force* F =500 *N is applied to a cord wrapped around it, causing the disk to roll. Determine the linear and the angular velocities of the disk when it has turned one revolution.*

Solution

The distance travelled in one revolution by the center of mass of the disc is

$$s = 2\pi R = 2\pi \cdot 0.3 = 0.6\pi \text{ m}.$$

As the cord unwinds, the force F acts through a distance $2s = 2 \cdot 0.6\pi = 1.2\pi$ m.

The work done by disc is

$$A = \int_0^{2s} F ds = F \cdot 2s = 500 \cdot 1.2\pi = 1885 \text{ J}.$$

The change in kinetic energy is

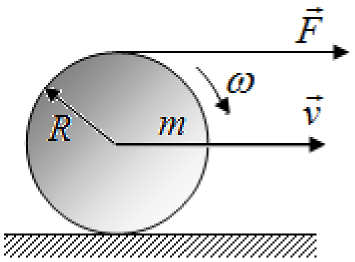
$$\Delta W_k = W_{k2} - W_{k1} = \Delta W_{kt} + \Delta W_{kr} = \left(W_{kt2} - W_{kt1}\right) + \left(W_{kr2} - W_{kr1}\right),$$

and with respect to the fact that the disc starts from rest

$$\Delta W_k = W_{kt2} + W_{kr2} = \frac{mv^2}{2} + \frac{I\omega^2}{2}.$$

Taking into account that $I = \dfrac{mR^2}{2}$ and $v = \omega R$, we obtain

$$\Delta W_k = \frac{mv^2}{2} + \frac{mR^2 v^2}{2 \cdot 2R^2} = \frac{3mv^2}{4}.$$



The work done changes the kinetic energy

$$A = \Delta W_k = \frac{3mv^2}{4},$$

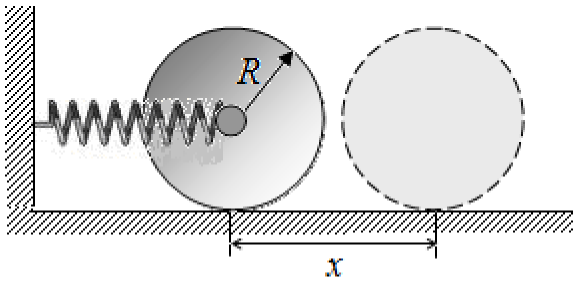
$$v = \sqrt{\frac{4A}{3m}} = \sqrt{\frac{4 \cdot 1885}{3 \cdot 100}} = 5 \text{ m/s}.$$

The angular velocity is equal to

$$\omega = \frac{v}{R} = \frac{5}{0.3} = 16.7 \text{ rad/s (clockwise).}$$

Problem 2.83

The 12 *kg homogenous cylindrical disk of radius* R =0.3 *m is given a clockwise angular speed of* 3 *rad/s with the spring unstretched. The spring constant is k* = 45 *N/m. If the disk rolls, how far will its center move to the right?*



Solution

If the disc rolls its initial kinetic energy is

$$W_k = W_{kt} + W_{kr} = \frac{mv^2}{2} + \frac{I\omega^2}{2} = \frac{m(\omega R)^2}{2} + \frac{mR^2\omega^2}{2 \cdot 2} = 0.75mR^2\omega^2.$$

The moving disc stretches the spring which potential energy is $W_p = \dfrac{kx^2}{2}$.

From the law of conservation of energy, $W_k = W_p$,

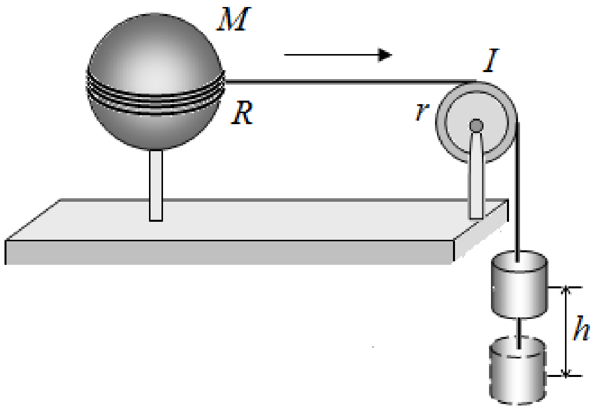
$$0.75mR^2\omega^2 = \frac{kx^2}{2}.$$

$$x = \sqrt{\frac{1.5mR^2\omega^2}{k}} = \sqrt{\frac{1.5 \cdot 12 \cdot 0.3^2 \cdot 3^3}{45}} = 0.57 \text{ m.}$$

Problem 2.84

A uniform spherical shell of mass $M = 21$ kg and radius $R = 1.5$ m rotates about a vertical axis on frictionless bearings. A massless cord passes around the equator of the shell, over a pulley of rotational inertia $I = 1$ kg· m² and radius $r = 0.6$ m, and is attached to a small object of mass $m = 7$ kg. There is no friction on the pulley's axle; the cord does not slip on the pulley. What is the speed of the object after it falls a distance $h = 5$ m from rest? Use energy considerations.

Solution



According to the law of conservation of energy, the potential energy of descending load transforms into the kinetic energy of rotating motion of the pulley and spherical shell, and translational motion of the load.

$$mgh = \frac{mv^2}{2} + \frac{I_1 {\omega_1}^2}{2} + \frac{I\omega^2}{2},$$

where m and v are the mass and velocity of the load; $I_1 = \frac{2}{3}MR^2$ and ω_1 are the moment of inertia and the angular velocity of the spherical shell; I and ω are the moment of inertia and the angular velocity of the pulley.

The linear velocity of the load is equal to the linear velocity of the cord, which is related to the angular velocities of the pulley and the shell as

$$v = \omega_1 R = \omega r.$$

$$mgh = \frac{mv^2}{2} + \frac{1}{2}\cdot\frac{2}{3}MR^2\left(\frac{v}{R}\right)^2 + \frac{1}{2}\cdot I\left(\frac{v}{r}\right)^2 = \frac{mv^2}{2} + \frac{Mv^2}{3} + \frac{Iv^2}{2r^2}.$$

The speed of the load may be found from

$$mgh = v^2\left(\frac{m}{2} + \frac{M}{3} + \frac{I}{2r^2}\right),$$

as following

$$v = \sqrt{\frac{mgh}{(m/2)+(M/3)+\left(I/2r^2\right)}} = \sqrt{\frac{7\cdot 9.8\cdot 5}{(7/2)+(21/3)+\left(1/2\cdot 0.6^2\right)}} = 5.37 \text{ m/s}.$$

Problem 2.85

From what height above the bottom of the loop must the cyclist in the figure start in order to just make it around the loop of radius R *=3 m. The mass of the cyclist with the bicycle is* m *=75 kg, the mass of each wheel is* m_0 *= 1.5 kg. Assume that the wheels are hoops with the moment of inertia* $I = m_0 r^2$*, and there is no friction.*

Solution

To travel in a loop, the cyclist must have a net force acting on him that equals the centripetal force he needs to keep travelling in a circle of the given radius and speed. At the top of his path (point C) the cyclist barely stays in contact with the track. The normal force the track applies to the cyclist at the top is just about zero. The only force keeping him on his circular track is the force of gravity, which means that at the apex, the speed of the object has to be such that the centripetal force equals the object's weight to keep it going in a circle whose radius is the same as the radius of the loop. That means that if this is the force needed

$$ma_n = \frac{mv^2}{R} = mg\,.$$

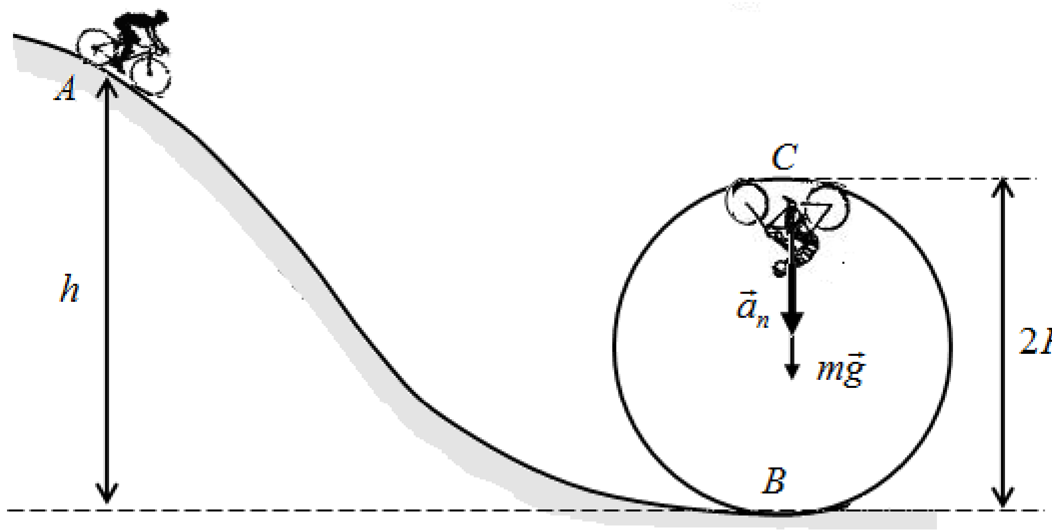
The minimum speed the cyclist needs at the top of the loop in order to keep going in a circle is

$$v = \sqrt{gR}\,.$$

According to the law of conservation of energy, the potential energy of the cyclist at his initial point *A* of motion ($W_p = mgh$) is equal to

$$W_p = W_{k1} + 2W_{k2} + W_{p1}\,,$$

where $W_{k1} = \dfrac{mv^2}{2}$ is the kinetic energy of the cyclist with the cycle at point *B*,



The kinetic energy of two rotating wheels is

$$W_{k2} = 2\frac{I\omega^2}{2} = \frac{m_0 r^2 v^2}{r^2} = m_0 v^2,$$

and the potential energy of the cyclist at point C is

$$W_p = 2mgR.$$

Hence,

$$mgh = \frac{mv^2}{2} + m_0 v^2 + 2mRg.$$

Taking into account that the velocity is $v = \sqrt{gR}$, let's write the conservation law:

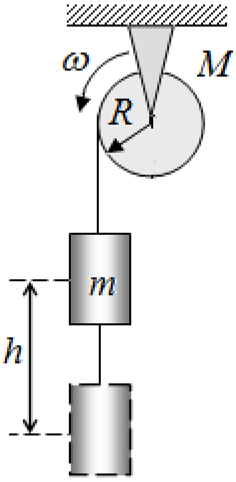
$$mgh = \frac{mgR}{2} + m_0 gR + 2mgR.$$

The cyclist has to start from the height

$$h = \frac{R}{2} + \frac{m_0}{m} gR + 2R = \frac{3}{2} + \frac{1.5 \cdot 3}{75} + 2 \cdot 3 = 7.56 \text{ m}.$$

Problem 2.86

The wheel is a solid disc of mass $M = 3$ *kg and radius* 40 *cm. The suspended block has mass* $m = 0.6$ *kg. If suspended block starts from rest and descends to a position* 1 *m lower, what is its speed when it is at this position?*



Solution

The work done on the system of the block and the wheel is due to the gravitational force $m\vec{g}$ acting on the hanging block. From the law of conservation of energy, the change in potential energy of the block

$$\Delta W_p = W_{p1} - W_{p2} = mg\Delta h$$

must be equal to the change in kinetic energy of the system

$$\Delta W_k = W_{k2} - W_{k1} = \Delta W_{kt} + \Delta W_{kr} = \left(W_{kt2} - W_{kt1}\right) + \left(W_{kr2} - W_{kr1}\right),$$

where ΔW_{kt} is the change in translational kinetic energy of the block, and ΔW_{kr} is the change in rotational kinetic energy of the wheel.

The system begins from rest ($W_{kt1} = W_{kr1} = 0$), so we can write

$$mg\Delta h = W_{kt2} + W_{kr2} = \frac{mv^2}{2} + \frac{I\omega^2}{2},$$

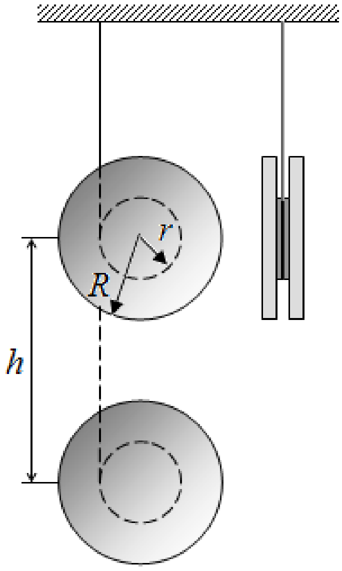
where v is the speed of the block in its final position. It is also the speed of the string at this instant as well as the speed of a point on the rim of the wheel at this instant. Therefore, $\omega = v/R$. In addition, the wheel is a solid disc, therefore, its moment of inertia is $I = \dfrac{MR^2}{2}$. Consequently,

$$mg\Delta h = \frac{mv^2}{2} + \frac{M\cancel{R^2}v^2}{2\cdot 2\cdot \cancel{R^2}} = \frac{mv^2}{2} + \frac{Mv^2}{4} = \frac{v^2\left(2m + M\right)}{4},$$

Solving for v, we find that

$$v = \sqrt{\frac{4mg\cdot\Delta h}{2m + M}} = \sqrt{\frac{4\cdot 0.6\cdot 9.8\cdot 1}{2\cdot 0.6 + 3}} = 2.37 \text{ m/s}.$$

Problem 2.87



The stepped disc consists of two discs of radius $R = 0.3$ *m and the mass* $M = 10$ *kg and one disc of radius* $r = 0.15$ *m and mass* $m = 5$ *kg. If the disk is released from rest, what is its angular velocity when the center of the disk has fallen* 1 *m*?

Solution

The change in potential energy of the disc is

$$\Delta W_p = W_{p1} - W_{p2} = (2M + m) g \Delta h .$$

The change of disc's kinetic energy (both translational and rotational) is

$$\Delta W_k = \Delta W_{kt} + \Delta W_{kr} = (W_{kt2} - W_{kt1}) + (W_{kr2} - W_{kr1}) \ .$$

Taking into account that the disc starts from rest its initial kinetic energy is zero

$$\Delta W_k = W_{kt2} + W_{kr2} = \frac{(2M + m) v^2}{2} + \frac{I\omega^2}{2} .$$

The moment of inertia of the stepped disc is

$$I = 2I_1 + I_2 = 2 \cdot \frac{MR^2}{2} + \frac{mr^2}{2} = 10 \cdot 0.3^2 + \frac{5 \cdot 0.15^2}{2} = 0.96 \text{ kg·m²}.$$

From the law of conservation of energy $\Delta W_p = \Delta W_k$, then

$$(2M + m) g \Delta h = \frac{(2M + m) v^2}{2} + \frac{I\omega^2}{2} .$$

The linear speed of the disc is $v = \omega \cdot r$

$$2(2M + m) g \Delta h = \left[(2M + m) r^2 + I \right] \omega^2$$

$$\omega = \sqrt{\frac{2(2M + m) g \Delta h}{(2M + m) r^2 + I}} = \sqrt{\frac{2 \cdot (2 \cdot 10 + 5) \cdot 9.8 \cdot 1}{(2 \cdot 10 + 5) \cdot 0.15^2 + 0.96}} = 17.9 \text{ rad/s}.$$

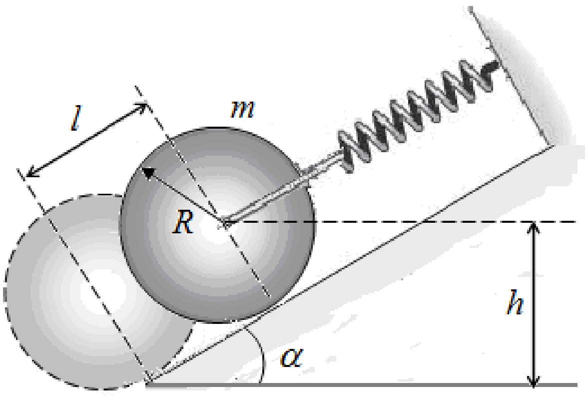
Problem 2.88

A sphere of mass M = 10 *kg on an incline of angle* $\alpha = 30°$, *is attached to a spring of constant k* = 100 *N/m. The spring is not stretched. Find the speed of the sphere when it has rolled a distance l* = 0.2 *m down the incline* ($g \approx 10$ *m/s*^{2}).

Solution

Due to the absence of friction the system examined in this problem is isolated, hence, we may use the law of conservation of energy.

As the object rolls distance l down the incline, it loses gravitational potential energy while it gains both linear and rotational kinetic energy. Moreover, the spring gains spring potential energy. Thus we have



$$W_p = W_{k1} + W_{k2} + W_{ps},$$

$$mgh = \frac{mv^2}{2} + \frac{I\omega^2}{2} + \frac{kx^2}{2},$$

where $h = l\sin\alpha$ and $I = \frac{2mR^2}{5}$ is the moment of inertia of a sphere about the axis through the centre of mass.

Since the object rolls without slipping, the angular velocity ω is related to the linear velocity v as $\omega = v/R$.

The equation of the conservation law after the substitution takes on form

$$mgl\sin\alpha = \frac{mv^2}{2} + \frac{\cancel{2}m\cancel{R^2}v^2}{\cancel{2}\cdot 5\cancel{R^2}} + \frac{kl^2}{2},$$

$$v^2 = \frac{10}{7m}\left(mgl\sin\alpha - \frac{kl^2}{2}\right).$$

Solving for v we get

$$v = \sqrt{\frac{mgl\sin\alpha - 0.5kl^2}{0.7m}} = \sqrt{\frac{10\cdot 10\cdot 0.2\cdot 0.5 - 0.5\cdot 100\cdot 0.04}{0.7\cdot 10}} = 1.14 \text{ m/s}.$$

Problem 2.89

Energy of 500 *J is spent in increasing the speed of fly-wheel from* 60 *rev/min to* 360 *rev/min. Find the moment of inertia of the wheel.*

Solution

The speed of rotation was changed from $n_1 = 30$ rev/min = 0.5 rev/s to $n_2 = 360$ rev/min = 6 rev/s, consequently, the kinetic energy of the wheel was changed. According to the Work-Energy Theorem, the work is a quantitative measure of changes in kinetic energy. Hence the work is equal to the increment of the kinetic energy of the wheel:

$$A = \Delta W_k = W_{k2} - W_{k1}.$$

Taking into account that $\omega = 2\pi n$, we can write the equation in the following form

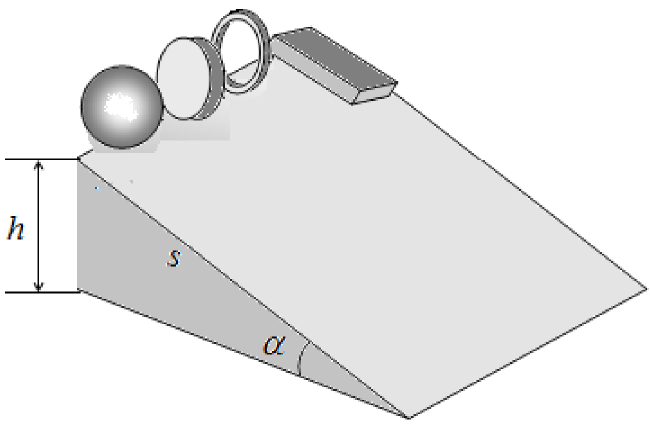
$$A = \Delta W_k = W_{k2} - W_{k1} = \frac{I\omega_2^2}{2} - \frac{I\omega_1^2}{2} = \frac{4 \cdot I \cdot \pi^2}{2}\left(n_2^2 - n_1^2\right).$$

Substituting given data, we obtain

$$I = \frac{A}{2 \cdot \pi^2 \cdot \left(n_2^2 - n_1^2\right)} = \frac{500}{2\pi^2\left(36 - 0.25\right)} = 4.4 \text{ kg}\cdot\text{m}^2.$$

Problem 2.90

Find the linear velocities and accelerations of the centers of sphere, disc and hoop that roll down an inclined plane without slipping. The incline of height h = 1 *m makes an angle of* 30° *with the horizontal. The initial velocity of all objects* $v_0 = 0$*. Compare calculated velocities and accelerations with the velocity and acceleration of the box, which slides from this incline without friction.*



Solution

Initially, the each object has the gravitational potential energy only.

When it reaches the bottom of the ramp, this potential energy has been converted to the translational and rotational kinetic energy according to the law of conservation of mechanical energy. Taking into account that the moments of inertia of the objects mentioned above are $I_{sp} = \frac{2}{5}mR^2$, $I_{disc} = \frac{1}{2}mR^2$, $I_{hoop} = mR^2$, and the angular speed of rolling objects is related to the liner speed according to $\omega = \frac{v}{R}$, we apply the law of conservation of energy $W_p = W_k$.

$$W_p = W_k = \begin{cases} W_{k_{sp}} = \dfrac{mv^2}{2} + \dfrac{\cancel{2} m \cancel{R^2} v^2}{5 \cdot \cancel{2} \cdot \cancel{R^2}} = 0.7mv^2 & \Rightarrow \quad v = \sqrt{\dfrac{gh}{0,7}} = 3.74\,\text{m/s}, \\ W_{k_{disc}} = \dfrac{mv^2}{2} + \dfrac{m \cancel{R^2} v^2}{2 \cdot 2 \cdot \cancel{R^2}} = 0.75mv^2 & \Rightarrow \quad v = \sqrt{\dfrac{gh}{0,75}} = 3.61\,\text{m/s}, \\ W_{k_{hoop}} = \dfrac{mv^2}{2} + \dfrac{m \cancel{R^2} v^2}{2 \cdot \cancel{R^2}} = mv^2 & \Rightarrow \quad v = \sqrt{gh} = 3.13\,\text{m/s}, \\ W_{k_{box}} = \dfrac{mv^2}{2} & \Rightarrow \quad v = \sqrt{2gh} = 4.43\,\text{m/s}. \end{cases}$$

Accelerated motion of each object is described by the following kinematic equations

$$\begin{cases} s = v_0 t + \dfrac{at^2}{2}, \\ v = v_0 + at. \end{cases}$$

Solving the system for a gives

$$a = \frac{v^2 - v_0^2}{2s}.$$

The initial velocities are $v_0 = 0$, the final velocities we have found above, the travelled distance is $s = h/\sin\alpha$, where h is the height of incline. The accelerations of the objects are

$$a = \frac{v^2 \cdot \sin\alpha}{2h} = \begin{cases} a_{sp} = \dfrac{g \cdot \sin\alpha}{0.7 \cdot 2} = \dfrac{9.8 \cdot 0.5}{1.4} = 3.5\,\text{m/s}^2, \\ a_{disc} = \dfrac{g \cdot \sin\alpha}{0.75 \cdot 2} = \dfrac{9.8 \cdot 0.5}{1.5} = 3.27\,\text{m/s}^2, \\ a_{hoop} = \dfrac{g \cdot \sin\alpha}{2} = \dfrac{9.8 \cdot 0.5}{2} = 2.45\,\text{m/s}^2, \\ a_{box} = \dfrac{\cancel{2} \cdot g \cdot \sin\alpha}{\cancel{2}} = 9.8 \cdot 0,5 = 4.9\,\text{m/s}^2. \end{cases}$$

Problem 2.91

The wheel during the time $t = 60$ s of decelerated motion diminished the frequency of rotation from $n_1 = 5$ rev/s to $n_2 = 3$ rev/s. Find the amount of revolutions N made during this time period, an angular acceleration ε of the wheel, the braking torque M and the work of braking force A. The wheel is the hoop of mass $m = 1$ kg and radius $R = 0.2$ m.

Solution

Decelerated motion of the wheel is described by the following kinematic equations

$$\begin{cases} 2\pi N = 2\pi n_1 t - \dfrac{\varepsilon t^2}{2}, \\ 2\pi n_2 = 2\pi n_1 - \varepsilon t. \end{cases}$$

An angular acceleration is

$$\varepsilon = \frac{2\pi(n_1 - n_2)}{t} = \frac{2\pi(5-3)}{60} = 0.21 \text{ rad/s}^2.$$

The amount of revolutions is

$$N = n_1 t - \frac{\varepsilon t^2}{4\pi} = 5 \cdot 60 - \frac{0.21 \cdot 60^2}{4\pi} = 240.$$

The moment of inertia of the hoop respectively the centre of mass equals

$$I = mR^2 = 1 \cdot 0.2^2 = 0.04 \text{ kg·m}^2.$$

Using the Newton's 2nd Law for rotation, the braking torque may be found as

$$M = I\varepsilon = 0,04 \cdot 0,21 = 8,4 \cdot 10^{-3} \text{ N·m}.$$

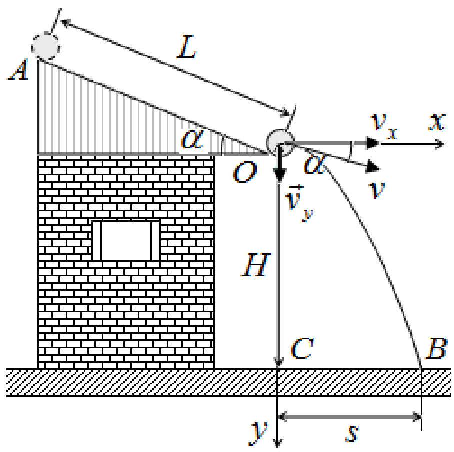
The work of braking force is determined by the increment of the kinetic energy of the rotating wheel

$$A = \Delta W_k = W_{k2} - W_{k1} = \frac{I\omega_2^2}{2} - \frac{I\omega_1^2}{2} = \frac{4\pi^2 I}{2}\left(n_2^2 - n_1^2\right) =$$

$$= 2 \cdot I\pi^2\left(n_2^2 - n_1^2\right) = 2 \cdot 0.04 \cdot \pi^2\left(3^2 - 5^2\right) = -12.63 \text{ J}.$$

Problem 2.92

A solid cylinder of radius R = 10 cm and mass m = 10 kg starts from rest and rolls without slipping a distance L = 5 m down a roof that is inclined at the angle $\alpha = 35°$. a) What is the angular speed of the cylinder about its center as it leaves the roof? b) The roof's edge is at height H = 10 m. How far horizontally from the roof's edge does the cylinder hit the level ground?



Solution

a) At the initial point of motion (point A) the cylinder has potential energy (W_p). It starts from rest, and then gains kinetic energy (W_k) as it rolls down the roof. The kinetic energy consists of the kinetic energy of translational (W_{kt}) and rotational (W_{kr}) motions.

$$W_p = W_k = W_{kt} + W_{kr},$$

$$mgH = \frac{mv^2}{2} + \frac{I\omega^2}{2},$$

$$mg \cdot L\sin\alpha = \frac{m(\omega R)^2}{2} + \frac{1}{2}\frac{mR^2\omega^2}{2} = \frac{3mR^2\omega^2}{4}$$

$$\omega = \sqrt{\frac{4gL \cdot \sin\alpha}{3R^2}} = \sqrt{\frac{4 \cdot 9.8 \cdot 5 \cdot \sin 35°}{3 \cdot 0.01}} = 61.2 \text{ rad/s}.$$

b) The linear velocity at which the cylinder leaves the roof is

$$v = \omega R = 61.2 \cdot 0.1 = 6.12 \text{ m/s}.$$

This velocity directed at the angle α below the positive direction of x-axis, hence, the horizontal component of the velocity is

$$v_x = v \cdot \cos\alpha = 6.12 \cdot \cos 35° = 5 \text{ m/s}.$$

The vertical component of the velocity is

$$v_y = v \cdot \sin\alpha = 6.12 \cdot \sin 35° = 3.51 \text{ m/s}.$$

The time of the cylinder vertical motion may be found from equation

$$H = v_y \cdot t + \frac{gt^2}{2},$$

$$gt^2 + 2v_y t - 2H = 0,$$

$$t_{1,2} = \frac{-2v_y \pm \sqrt{(2v_y)^2 + 8 \cdot g \cdot H}}{2g} = \frac{-2 \cdot 3.51 \pm \sqrt{(2 \cdot 3.51)^2 + 8 \cdot 9.8 \cdot 10}}{2 \cdot 9.8}.$$

Taking the positive root of this equation we obtain the time of vertical motion $t = 0.73$ s (the second negative root is physically meaningless).

Now, the desired distance is

$$s = v_x t = 5 \cdot 0.73 = 3.65 \text{ m}.$$

Problem 2.93

A tennis ball, starting from rest, rolls down the hill from the height h =2 m. At the end of the hill the ball becomes air-borne, leaving at an angle of $30°$ with respect to the ground. Treat the ball as a thin-walled spherical shell, and determine the range L.

Solution

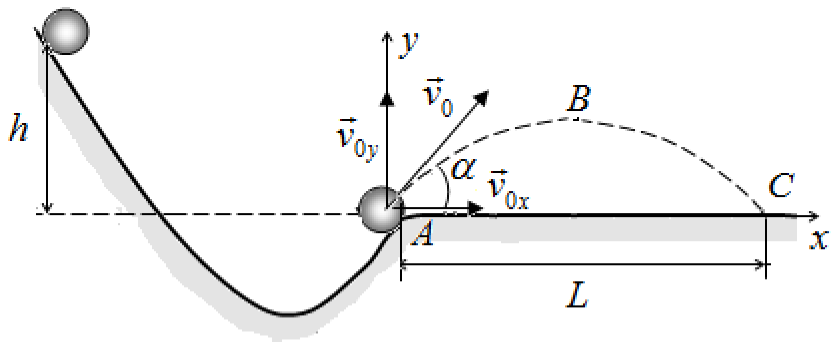
When the tennis ball starts from rest, its total mechanical energy is in the form of gravitational potential energy $W_p = mgh$ (we take $h = 0$ at the height where the ball becomes airborne). Just before the ball becomes airborne, its mechanical energy is in the form of rotational kinetic energy and translational kinetic energy. At this instant its total kinetic energy is $W_k = \frac{mv_0^2}{2} + \frac{I\omega_0^2}{2}$. If we treat the tennis ball as a thin walled spherical shell of mass m and radius R, its moment of inertia is $I = \frac{2mR^2}{3}$. The angular speed related to the linear speed as $\omega = v/R$. If we take into account that the ball rolls down the hill without slipping, its total kinetic energy can be written as

$$W_k = \frac{mv_0^2}{2} + \frac{2m \cdot R^2 \cdot v_0^2}{3 \cdot 2 \cdot R^2} = \frac{5mv_0^2}{6}.$$

Therefore, from conservation of mechanical energy, we have $W_p = W_k$, or

$$mgh = \frac{5mv_0^2}{6},$$

$$v_0 = \sqrt{\frac{6gh}{5}}.$$



The y-component of the ball's speed v_y depends on time as $v_y = v_{0y} - gt = v_0 \cdot \sin\alpha - gt$, and at the top point of the trajectory (point B) is zero.

$$v_y = v_0 \cdot \sin\alpha - gt = 0.$$

Hence the time of ball's flight to the point B is

$$t = \frac{v_0 \cdot \sin\alpha}{g}.$$

The total time of flight from A to C is $t_0 = 2t = \dfrac{2v_0 \cdot \sin\alpha}{g}$.

The range of the tennis ball is given by

$$L = v_{0x} t_0 = \frac{v_0 \cdot \cos\alpha \cdot 2v_0 \cdot \sin\alpha}{g} = \frac{v_0^2 \cdot \sin 2\alpha}{g} = \frac{6gh \cdot \sin 2\alpha}{5g} = \frac{6h \cdot \sin 2\alpha}{5}.$$

Substitution of the numerical values gives

$$L = \frac{6 \cdot 2 \cdot \sin 60°}{5} = 2.08 \text{ m}.$$

Problem 2.94

A billiard ball that is initially at rest is given a sharp blow by a cue stick. The force is horizontal and is applied at a distance $h = 2R/3$ below the centerline. The speed of the ball just after the blow is v_0 and the coefficient of kinetic friction between the ball and the billiard table is μ. Find a) the angular velocity just after the blow; b) velocity once rolls without slipping; c) kinetic energy just after the hit; and d) the work of the friction force.

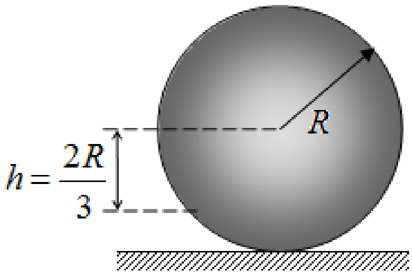
Solution

a) The moment of inertia of the billiard ball (the solid sphere) about the axis passing through the ball's center is $I = \dfrac{2mR^2}{5}$. Its angular momentum is

$$L = I \cdot \omega_0 = \frac{2mR^2}{5} \cdot \omega_0. \qquad (1)$$

On the other hand, an angular momentum is

$$L = p \cdot h = mv_0 \cdot \frac{2R}{3}. \qquad (2)$$



Equating expressions (1) and (2) gives

$$\frac{2mR^2}{5} \cdot \omega_0 = mv_0 \cdot \frac{2R}{3},$$

and magnitude of initial angular velocity is

$$\omega_0 = \frac{5v_0}{3R}.$$

Since the force is applied below the center line, the spin is backward, i. e., the ball will slow down, and its angular velocity is $\omega_0 = -\frac{5v_0}{3R}$.

b) The friction force for the object on the horizontal surface is

$$F_{fr} = \mu \cdot N = \mu \cdot mg.$$

The torque of friction force is

$$M = F_{fr} \cdot R = \mu \cdot mg \cdot R.$$

Then, from Newton's 2nd Law for rotation $M = I\varepsilon$, an angular acceleration is

$$\varepsilon = \frac{M}{I} = \frac{\mu \cdot mg \cdot R \cdot 5}{2mR^2} = \frac{5\mu \cdot g}{2R}.$$

When the ball begins to move according to Newton's 2nd Law

$$ma = F_{fr} = \mu \cdot N = \mu \cdot mg.$$

Hence, the acceleration of the ball is

$$a = \mu \cdot g,$$

and its linear velocity is

$$v = v_0 - at = v_0 - \mu \cdot g \cdot t. \qquad (3)$$

An angular velocity using the results obtained in the part (a) is

$$\omega = \omega_0 + \varepsilon \cdot t = -\frac{5v_0}{3R} + \left(\frac{5\mu \cdot g}{2R}\right)t.$$

Since $v = \omega R$, the linear velocity is

$$v = -\frac{5v_0}{3R} \cdot R + \left(\frac{5\mu \cdot g}{2R}\right) \cdot R \cdot t = -\frac{5v_0}{3} + \frac{5\mu \cdot g}{2} \cdot t. \qquad (4)$$

c) Equating expressions (3) and (4) for the linear velocity gives

$$v_0 - \mu \cdot g \cdot t = -\frac{5v_0}{3} + \frac{5\mu \cdot g}{2} \cdot t .$$

$$v_0 + \frac{5v_0}{3} = \mu \cdot g \cdot t + \frac{5\mu \cdot g}{2} t = \left(1 + \frac{5}{2}\right) \cdot \mu \cdot g \cdot t = \frac{7}{2} \cdot \mu \cdot g \cdot t ,$$

$$\frac{8v_0}{3} = \frac{7\mu \cdot g \cdot t}{2} ,$$

$$t = \frac{16 \cdot v_0}{21 \cdot \mu \cdot g} . \tag{5}$$

Plugging (5) in (3), we finally obtain the linear velocity of the ball

$$v = v_0 - kgt = v_0 - kg \cdot \frac{16 \cdot v_0}{21 \cdot kg} = \frac{5v_0}{21} = 0.238 v_0 .$$

c) Kinetic energy of the ball is the sum of kinetic energies of translational and rotational motions.

$$W_0 = \frac{mv_0^2}{2} + \frac{I\omega_0^2}{2} = \frac{mv_0^2}{2} + \frac{1}{2}\left(\frac{2mR^2}{5}\right)\left(\frac{5v_0^2}{3R^2}\right) = \frac{19}{18} mv_0^2 = 1.056 \cdot mv_0^2 .$$

$$W = \frac{mv^2}{2} + \frac{I\omega^2}{2} = \frac{mv^2}{2} + \frac{1}{2}\left(\frac{2mR^2}{5}\right)\left(\frac{v^2}{R^2}\right) = 0.7mv^2 = 0.7m\left(0.238v_0\right)^2 = 0.4mv_0^2 .$$

$$\Delta W = W - W_0 = 1.056 \cdot mv_0^2 - 0.4mv_0^2 = 1.016mv_0^2 .$$

d) The work of the friction force is equal to the change in kinetic energy

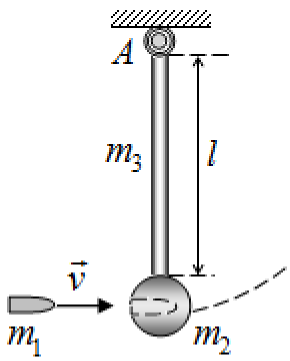
$$A_{fr} = -\Delta W = -1.016mv_0^2 .$$

Problem 2.95

A 3 g bullet is fired into a 0.8 kg block attached to the end of a 0.6 m non-uniform rod of mass 0.5 kg. The block-rod-bullet system then rotates in the plane of the figure about a fixed axis at A. The rotational inertia of the rod alone about that axis at A is 0.06 kg·m². Treat the block as a particle. (a) What is the rotational inertia of the block-rod-bullet system about point A? (b) If the angular speed of the system about the point A just after impact is 4.5 rad/s, what is the bullet's speed just before impact?

Solution

If the particle of the constant mass is moving at a distance r from the chosen origin it has the angular momentum equaled to $\vec{L} = [\vec{r}, \vec{p}]$. The magnitude of the angular momentum of the bullet of mass m_1 about a point A is



$$L = pl = m_1 vl .$$

After the bullet embeds itself in the block, the whole thing consisting of the rod and block with the bullet inside it rotates. The total moment of inertia is

$$I = I_1 + I_2 + I_3 = m_1 l^2 + m_2 l^2 + I_3 = (0.003 + 0.8)0.6^2 + 0.06 = 0.35 \text{ kg·m²}.$$

Initially, the single object that moves is the bullet, therefore, the angular momentum is

$$L_{initial} = L_1 = m_1 vl .$$

Finally, the rod with the block and the bullet are rotating about the point A, thus the angular moment of the system is

$$L_{final} = I\omega .$$

According to the law of conservation of angular momentum

$$L_{initial} = L_{final} .$$

$$m_1 vl = I\omega .$$

The velocity of the bullet before impact was

$$v = \frac{I\omega}{m_1 l} = \frac{0.35 \cdot 4.5}{0.003 \cdot 0.6} = 875 \text{ m/s}.$$

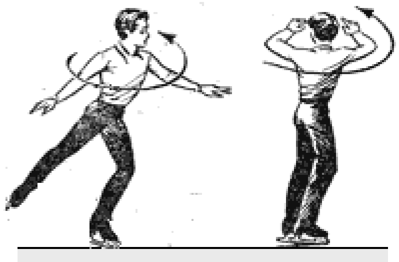
Problem 2.96

An ice skater begins a spin by rotating at an angular velocity of 2.2 *rad/s with both arms and one leg outstretched. At that time her moment of inertia is 0.52 kg·m². Then he brings his arms up over her head and her legs together, reducing his moment of inertia by 0.21 kg·m². At what angular velocity will he then spin?*

Solution

Because there are no acting external torques (any friction is ignored here), angular momentum is conserved and we can write that

$$I_1\omega_1 = I_2\omega_2 .$$



In this case the skater's moment of inertia has decreased and so his angular velocity will increase.

We find

$$\omega_2 = \frac{I_1\omega_1}{I_2} = \frac{0.52 \cdot 2.2}{0.21} = 5.4 \text{ rad/s}.$$

Problem 2.97

A 50 *kg student is spinning on the merry-go-round that has a mass of* 100 *kg and a radius of 2 m. She walks from the edge of merry-go-round towards the center. If the angular speed of the merry-go-round is initially* 2 *rad/s^2, what is its angular speed when the student reaches a point* 0.5 *m from center?*

Solution

Because there is no external torques, the angular momentum of the system (merry-go-round plus student) is conserved, $L = L'$,

$$L_m + L_s = L'_m + L'_s .$$

Determine the moments of inertia. Treat the merry-go-round as a solid disc, and treat the student as a point mass.

$$I_m = I'_m = \frac{MR^2}{2}, \quad I_s = mR^2, \quad I'_s = mr^2.$$

$$I_m\omega_1 + I_s\omega_1 = I'_m\omega_2 + I'_s\omega_2.$$

$$\omega_1\left(\frac{MR^2}{2} + mR^2\right) = \omega_2\left(\frac{MR^2}{2} + mr^2\right)$$

Substitute the values into the equations and solve for ω_2

$$\omega_2 = \omega_1\frac{MR^2 + 2mR^2}{MR^2 + 2mr^2} = \omega_1\frac{M + 2m}{M + 2m\left(\dfrac{r}{R}\right)^2} = 2\cdot\frac{100 + 2\cdot 50}{100 + 2\cdot 50\left(\dfrac{0.5}{2}\right)^2} = 3.76 \text{ rad/s}.$$

Problem 2.98

A merry-go-round of radius $R = 6$ *m with nearly frictionless bearings and a moment of inertia* $I = 3000$ *kg·m² is turning at* $n_1 = 3$ *rpm when the motor is turned off. If there were* 10 *children of* $m = 25$ *kg average mass initially out at the edge of the carousel and they all move into the center and huddle* $r = 1$ *m from the axis of rotation, find the angular velocity of the carousel. If then the brakes are applied, find the torque required to stop the carousel in* 15 *s.*

Solution

Before the brakes are applied there are no external torques acting on the carousel (friction is absent in the bearings) so that we know angular momentum is conserved. Therefore, we can first write expressions for the initial and final angular momentum and then equate them to solve for the final angular speed. We have

$$L_1 = I_1\omega_1$$

Initial moment of inertia I_1 consists of the moment of inertia of the carrousel I and the moments of inertia of the children at the edge

$$I_1 = I + N\cdot I_0 = I + N\cdot mR^2 = 3000 + 10\cdot 25\cdot 6^2 = 12000 \text{ kg·m²},$$

treating the children as point masses at the edge of the carousel.

The angular speed is

$$\omega_1 = 2\pi n_1 = \frac{2\pi \cdot 3}{60} = \frac{\pi}{10} = 0.314 \text{ rad/s.}$$

The initial angular moment is

$$L_1 = I_1 \cdot \omega_1 = 12000 \cdot 0.314 = 3768 \text{ kg·m}^2\text{/s.}$$

Final moment of inertia I_2 consists of the moment of inertia of the carrousel I and the moments of inertia of the children at the distance r from the axis of rotation:

$$I_2 = I + N \cdot I_0 = I + N \cdot mr^2 = 3000 + 10 \cdot 25 \cdot 1^2 = 3250 \text{ kg·m}^2.$$

Using conservation of angular momentum, we then can write

$$L_1 = L_2 = I_2\omega_2 .$$

So that the final angular speed is

$$\omega_2 = \frac{L_1}{I_2} = \frac{3768}{3250} = 1.16 \text{ rad/s.}$$

From kinematics the angular speed for decelerated motion depends on time as

$$\omega = \omega_0 - \varepsilon t ,$$

where ω_0 and ω are the initial and final angular speeds, respectively. For this problem, the braking begins at the speed $\omega_0 = \omega_2$ and ends when $\omega = 0$. Hence $0 = \omega_2 - \varepsilon t$,

$$\varepsilon = \frac{\omega_2}{t} = \frac{1.16}{15} = 0.077 \text{ rad/s}^2.$$

Using Newton’s 2nd Law for rotation and substituting the values we obtain

$$M = I_2\varepsilon = 3250 \cdot 0.077 = 250.25 \text{ N·m.}$$

Problem 2.99

A fly-wheel begins to rotate with an angular acceleration ε = 0.4 *rad/s*^{2} *and after* t_1 = 10 *s has the kinetic energy* W_k = 80 *J. Find the angular momentum of the fly-wheel after* t_2 = 30 *s since the beginning of the rotation.*

Solution

After t_1 = 10 seconds of rotation ($\omega_0 = 0$) an angular velocity of fly-wheel is

$$\omega_1 = \omega_0 + \varepsilon t_1 = \varepsilon t_1 = 0.4 \cdot 10 = 4 \text{ rad/s}.$$

Since kinetic energy for rotating body is

$$W_k = \frac{I\omega_1^2}{2},$$

and moment of inertia of the fly-wheel is

$$I = \frac{2W_k}{\omega_1^2} = \frac{2 \cdot 80}{16} = 10 \text{ kg·m}^2.$$

After t_2 = 20 seconds of rotation an angular velocity is

$$\omega_2 = \varepsilon t_2 = 0.4 \cdot 30 = 12 \text{ rad/s}.$$

The angular momentum on this instant of time is equal to

$$L = I\omega_2 = 10 \cdot 12 = 120 \text{ kg·m}^2\text{·s}^{-1}.$$

Problem 2.100

The hoop of radius R = 1 *m is hooked and may oscillate in vertical plane. Find the period of hoop's oscillations.*

Solution

The hoop in this problem is the compound pendulum, hence its period of oscillations is

$$T = 2\pi\sqrt{\frac{I}{mgx}},$$

where m is the hoop mass, I is the moment of inertia respectively the pivot point, and x is the distance between the pivot point and the centre of mass.

The moment of inertia may be determined using the parallel axes theorem (Huygens–Steiner theorem): moment of inertia of a rigid body about any axis is sum of the body's moment of inertia about the parallel axis passing through the object's centre of mass and the product of the mass and the perpendicular distance between the two axes. Hence,

$$I = I_0 + mx^2 = mR^2 + mR^2 = 2mR^2 .$$

The distance $x = R$ .The period of oscillations is

$$T = 2\pi\sqrt{\frac{2\cancel{m}R^{\cancel{2}}}{\cancel{m}g\cancel{R}}} = \sqrt{\frac{2R}{g}} = \sqrt{\frac{2\cdot 0.5}{9.8}} = 0.32\,\text{s}.$$

CONTROL PROBLEMS

1. A 4 kg object has a velocity of $3\vec{i}$ m/s at one instant. Eight seconds later, its velocity is $\left(8\vec{i}+10\vec{j}\right)$ m/s. Assuming the object was subject to a constant net force, find a) the force, b) the components of the force, and c) its magnitude. [$F_x = 2.5$ N, $F_y = 5$ N, $F = 5.6$ N, α=63.4º]

2. While two forces act on it, a particle of mass m is to move with constant velocity. One of the forces is $F_1 = \left(2\vec{i}-6\vec{j}\right)$ N. What is the other force? [$\vec{F}_2 = -\vec{F}_1 = \left(-2\vec{i}+6\vec{j}\right)$N.]

3. A 3 kg mass undergoes an acceleration given by $\vec{a} = \left(2\vec{i}+5\vec{j}\right)$ m/s². Find the resultant force $\vec{F}$ and its magnitude. [$\vec{F} = \left(6\vec{i}+15\vec{j}\right)$N, $F = 16.16$ N]

4. Three forces $\vec{F}_1 = \left(2\vec{i}-5\vec{j}+2\vec{k}\right), \vec{F}_2 = \left(-4\vec{i}+8j+\vec{k}\right),\ \vec{F}_3 = \left(5\vec{i}+2\vec{j}-5\vec{k}\right)$ act on a particle with mass 6 kg. The forces are in Newtons. a) What is the net force vector? b) What is the magnitude of the net force? c) What is the acceleration vector? d) What is the magnitude of the acceleration vector? [$\vec{F} = \left(3\vec{i}+5\vec{j}-2\vec{k}\right)$, $F = 6.2$ N, $\vec{a} = \left(0.5\vec{i}+0.83\vec{j}-0.33\vec{k}\right), a = 1.03$ m/s²]

5. Three forces $\vec{F}_1 = (25\,\text{N}, 42.5°)$, $\vec{F}_2 = (15.5\,\text{N}, 215°)$, and $\vec{F}_3 = (20.5\,\text{N}, 155°)$ accelerate an 8.75 kg mass. What is the net force acting on the mass? What is the magnitude and direction of the mass's acceleration? What would have to be the magnitude and direction of a fourth force $\vec{F}_4$ so that the acceleration of the mass would be zero?
[$F = 21.04$ N, $\alpha = \arctan\left(F_y/F_x\right) = 52.4°$, $\vec{F} = (21\,\text{N}, 127.6°)$; $a = 2.4$ m/s², $127.6°$; $\vec{F}_4 = \left(21\,\text{N}, 307.6°\right)$]

6. The apparent weight of a person in an elevator is 0.875 of his actual weight. What is the acceleration (including the direction) of the elevator? [$a = 1.23$ m/s², downwards]

7. What is the magnitude of the average force required to stop the 1500 kg car in 8.0 s if the care is traveling at 95 km/h? [$F = 4948$ N]

8. A 210 kg motorcycle accelerates from 0 to 90 km/h in 6 s. a) What is the magnitude of the motorcycle's constant acceleration? b)What is the magnitude of the net force causing the acceleration? [$a = 4.17$ m/s², $F = 875$ N]

9. A 4.53 kg block initially at rest is pulled to the right along a horizontal, frictionless surface by a constant, horizontal force of 13 N. Find the speed of the block after it has moved 2.72 m horizontally. [$v = 3.95$ m/s]

10. A 3.81 kg block initially at rest is pulled to the right along a horizontal surface by a constant, horizontal force of 17.8 N. The coefficient of kinetic friction is 0.12. Find the speed of the block after it has moved 3.57 m horizontally. [$v = 5$ m/s]

11. The Earth has a mass $m = 5.98 \cdot 10^{24}$ kg and the Sun has a mass $M = 1.99 \cdot 10^{30}$ kg. They are separated, center to center, by R= 150 million km. What is the size of the gravitational force acting on the Earth due to the Sun? [$F = 5.3 \cdot 10^{32}$ N]

12. The distance between the centres of the Earth and the Moon is $3.85 \cdot 10^{8}$ m. The Moon has a mass which is only 1.29% that of Earth. Where would a satellite have to be placed to feel no net gravitational pull from the Earth and the Moon? [$R = 3.46 \cdot 10^{8}$ m]

13. A block of mass m_1= 3.7 kg on a frictionless inclined plane of angle $\alpha = 30°$ is connected by a cord over a massless, frictionless pulley to a second block of mass m_2= 2.3 kg hanging vertically. What are a) the magnitude of the acceleration of each block and b) the direction of the acceleration of m_2? c) What is the tension in the cord? [$a = 0.735$ m/s²; downwards; $T = 20.8$ N]

14. Two boxes, mass 4 kg and 5 kg are set next one another on a frictionless surface. The first box is pushed with a force of 35 N. What normal forces does the second box exert on the first box during this process? [$F = 19.5$ N]

15. A 40-N object requires 5 N to start moving over a horizontal surface. Find the coefficient of static friction. [$\mu = 0.13$]

16. A 12-N cart is moving on a horizontal surface with a coefficient of kinetic friction of 0.1. What force of friction must be overcome to keep the object moving at constant speed? [$F = 1.2$ N]

17. A 100 N block is on rough horizontal surface. If a horizontal force F = 40 N is applied and the block just start slide. What is the value of the static coefficient? [$\mu = 0.4$]

18. A 13-kg box is released on a 33° incline and accelerates down the incline at 0.2 m/s². Find the coefficient of kinetic friction.

[$\mu = \tan\alpha - \dfrac{a}{g \cdot \cos\alpha} = 0.625$]

19. A 205-kilogram log is pulled up a ramp by means of a rope that is parallel to the surface of the ramp. The ramp is inclined at 30° with respect to the horizontal. The coefficient of kinetic friction between the log and the ramp is 0.9, and the log has an acceleration of 0.8 m/s². Find the tension in the rope. [$T = 2730\,\text{N}$]

20.A box is sliding up an incline that makes an angle of 15° with respect to the horizontal. The coefficient of kinetic friction between the box and the surface of the incline is 0.18. The initial speed of the box at the bottom of the incline is 1.5 m/s. How far does the box travel along the incline before coming to rest? [$s = 0.265$ m]

21. It takes 45 N to make a 10 kg cart move at a constant speed. What force does it take to make the cart accelerate at 3.2 m/s² in the direction it is moving? [$F = 77\,\text{N}$]

22. A 5.5 box is under an upward force of 45 N which makes a 60° angle with its normal to the surface. Find the magnitude of the horizontal acceleration. What is the normal force? [$a = 7.1\,\text{m/s}^2$; $N = 31.4\,\text{N}$]

23. A ball having a mass of 4 kg is attached to a string 1 m long and is whirled in a vertical circle at a constant speed of 23 m/s. a) Determine the tension in the string when the ball is at the top of the circle. b) Determine the tension in the string when the ball is at the bottom of the circle. [$T_1 = 2076.8\,\text{N}$; $T_2 = 2155.2$ N]

24. A 0.4-kg ball, attached to the end of a horizontal cord, is rotated in a circle of radius 2m on a frictionless horizontal surface. If the cord will break when the tension in it exceeds 75 N, what is the maximum speed the ball can have? [$v = 19.36\,\text{m/s}$]

25. A car is travelling over the crest of a small semi-circular hill of radius R = 750 m. How fast would it have to be travelling for it to leave the ground? [$v = 85.8\,\text{m/s}$]

26. A rollercoaster is at the inside top of a circular loop of radius R = 150 m. How fast must the rollercoaster be going if it isn't to fall off? [$v \geq 38.4\,\text{m/s}$]

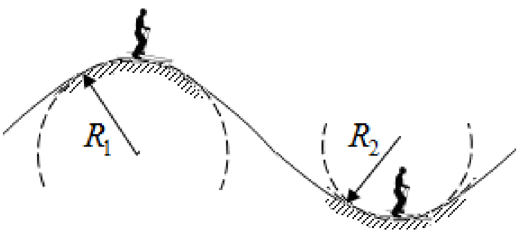
27. A bicyclist is riding on the banked curve of a circular velodrome. The radius of curvature for the bicyclist's present position is $R = 355$ m. The coefficient of static friction between the wheels and the path is $\mu = 0.35$. For which range of velocities will the bicyclist remain at the same height on the banked curve? [$18.7 \le v \le 58.3$ m/s]

28. A jet pilot takes his aircraft in a vertical loop. If the jet is moving at a speed of 1100 km/h at the lowest point of the loop: a) Determine the minimum radius of the circle so that the centripetal acceleration at the lowest point does not exceed $5.5g$ m/s²; b) Calculate the 68-kg pilot's effective weight (the force with which the seat pushes up on him) at the bottom of the circle; c) Calculate the 68-kg pilot's effective weight (the force with which the seat pushes up on him) at the top of the circle (assume the same speed) [$R > 1732$ m; $G_1 = 2998.8$ N; $G_2 = 4331.6$ N]

29. A 1400 kg car rounds a curve of 57 m banked at an angle of 14°. If the car is traveling at 98 km/h, will a friction force be required? If so, in what direction? [the friction force is required; $v = 27.2 > v_0 = 11.8$ m/s]

30. A coin is placed on a record that is rotating at 33.3 rpm. If the coefficient of static friction between the coin and the record is 0.1, how far from the center of the record can the coin be placed without having it slip off? [$r = 8$ cm]

31. An ice skater moving at 12 m s coasts to a halt in 95m on an ice surface. What is the coefficient of (kinetic) friction between ice and skates? [$\mu = 0.077$]



32. A moving skier on top of a circular hill of radius R_1= 62 m feels that his "weight" is only $3mg/8$. What would be the skier's apparent weight (in multiples of mg) at the bottom of the circular valley which has a radius R_2 = 43m? Neglect friction and air resistance. [$N_2 = 11.7mg$]

33. A 3-kg block starts from rest at the top of a $30°$-incline and slides 2 m down the incline in 1.5 s. Find a) the magnitude of the acceleration of the block; b) the coefficient of kinetic friction between the block and the plane; c) the frictional force acting on the block and d) the speed of the block after it has slid 2 m. [$a = 1.78$ m/s², μ=0.368, $F_{fr} = 9.4$N; $v = 2.67$ m/s]

34. A block is placed on a plane inclined at $35°$ relative to the horizontal. If the block slides down the plane with an acceleration of magnitude $g/3$, determine the coefficient of kinetic friction between block and plane. [$\mu = 0.293$]

35. What is the speed of a 1000-kilogram car that has a momentum of $2 \cdot 10^4$ kg·m/s? [$v = 20$ m/s]

36. A 2-kg body is initially traveling at a velocity of 40 m/s. If a constant force of 10 N is applied to the body in direction of its motion for 5 seconds, find the final speed of the body. [$v_2 = 65$ m/s]

37. A 5-N force imparts an impulse of 15 N·s to an object. Determine a period during which the force acted on the object. [$t = 3$ s]

38. An impulse of 30 N·s is applied to a 5-kg mass. If the mass had a speed of 100 m/s before the impulse, find its speed after the impulse. [$v_2 = 106$ m/s]

39. A 156 gram ball rolls across a pool table at 2 m/s when it collides with another ball of equal mass rolling in the opposite direction at 1.5 m/s. The first ball bounces backwards at 1 m/s. Find the momentum of the second ball. [$p = 0.234$ kg·m/s]

40. Two balls are involved in a completely inelastic collision. The first ball has a mass of 0.25 kg and an initial velocity of 9.27 m/s in the x-direction. The second ball has a mass of 0.50 kg with an initial velocity of 2.61 m/s in the opposite direction. What is the speed of the two balls after the collision in m/s? [$u = 0.35$ m/s]

41. Two objects A and B are involved in a totally elastic collision. The mass of A is 8.8 kg and the mass of B is 0.5 kg. The velocity of A is 4.5 m/s and the object B is at rest. If they collide elastically, what will be the final velocities of A and B in m/s? [$u_1 = 4.02$ m/s; $u_2 = 8.53$ m/s]

42. On a frictionless surface, a 6-kg rock approaches from the left at 3.5 m/s. It collides elastically with a 9-kg rock which is approaching from the right at 1.7 m/s. Find the final velocities of the rocks after collision. [u_1 =2.74 m/s; u_2 =2.46 m/s]

43. A spring has a force constant of 40 000 N/m. How far must it be stretched for its potential energy to be 8 J? [$x = 0.02$ m]

44. The 30-N box is at rest when the constant force F is applied. The force makes the angle 20° below horizontal (south of east). Two seconds later, the box is moving to the right at 20 m/s. Determine the applied force if a) the horizontal surface is smooth, and b) the coefficient of kinetic friction is $\mu = 0.1$. [$F_1 = 32.5\,\text{N}$, $F_2 = 37.1\,\text{N}$]

45. The force required to stretch a Hooke's-law spring varies from 0 to 63.4 N as we stretch the spring by moving one end 4.95 cm from its unstressed position. Find a) the force constant of the spring, and b) the work done in stretching the spring. [$k = 1280.8\,\text{N/m}$; $A = 1.6\,\text{J}$]

46. A 1.2-kg block and a 1.8-kg block are initially at rest on a frictionless, horizontal surface. When a compressed spring between the blocks is released, the 1.8-kg block moves to the right at 2 m/s. What is the speed of the 1.2-kg block after the spring is released? [$u_2 = 3$ m/s]

47. If the collision in previous question had been perfectly inelastic, what would have been the final velocity of the rocks? How much kinetic energy would have been lost in the collision? [u=0.38 m/s; $\Delta E_k = -48.7$J]

48. A 50-kg skater is travelling due east at 3 m/s. A 70-kg skater is moving due south at 7 m/s. They collide and hold on to one another after the collision. Determine the magnitude and direction of their velocity after the collision. Ignore the effects of friction [u=4.27 m/s; $\alpha = \arctan\left(u_y/u_x\right) = 79.98°$]

49. Nitrogen gas molecules, which have mass $4.65 \cdot 10^{-26}$ kg, are striking a vertical container wall at a horizontal velocity of positive 440 m/s. $5 \cdot 10^{21}$ molecules strike the wall each second. Assume the collisions are perfectly elastic, so each particle rebounds off the wall in the opposite direction but at the same speed. a) What is the change in momentum of each particle? b) What is the average force of the particles on the wall? [$\Delta p = 4.1 \cdot 10^{-23}$ kg·m/s; $F = N \cdot F_0 = 0.2\,\text{N}$]

50. A 0.075-kilogram arrow is fired horizontally. The bowstring exerts an average force of 65 N on the arrow over a distance of 0.90 meters. With what speed does the arrow leave the bow? [$v = 39.5$ m/s]

51. A skateboarder moving at 5.4 m/s along a horizontal section of a track that is slanted upward by 48° above the horizontal at its end, which is $h =$ 0.4 meters above the ground. When she leaves the track, she follows the characteristic path of projectile motion. Ignoring friction and air resistance, find the maximum height H to which she rises above the end of the track. [$H = 0.6$ m]

52. A student, starting from rest, slides down a water slide. On the way down, a kinetic frictional force (a nonconservative force) acts on her. The student has a mass of 71 kg, and the height of the water slide is 12 m. If the kinetic frictional force does −7400 J of work, how fast is the student going at the bottom of the slide? [v=11.44 m/s]

53. An astronaut of mass 82.2 kg is 31.2 m far out in space, with both the space ship and the astronaut at rest with respect to each other. Without a thruster, the only way to return to the ship is to throw his 0.502 kg wrench directly away from the ship. If he throws the wrench with a speed of 20 m/s, how many seconds does it take him to reach the ship? [t = 255.7 s = 4.26 min]

54. The magnitude of the momentum of a cat is p. What would be the magnitude of the momentum (in terms of p) of a dog having two times the mass of the cat if it had the same speed as the cat? What would be the magnitude of the momentum (in terms of p) of a dog having two times the mass of the cat if it had the same kinetic energy as the cat? [$p_2 = 2p_1 = 2p$; $p_2 = \sqrt{2} \cdot p_1 = \sqrt{2} \cdot p$]

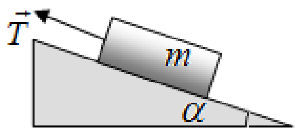
55. A slingshot is used to launch a 5-gram rock. The effective spring constant of the slingshot is 200 N/m. By what distance must the slingshot be stretched in order to launch the rock at 20 m/s? [x = 0.1 m]

56. A 2.5kg rock is dropped from a 350 m cliff. It hits the ground with a speed of 60 m/s. What percent of its mechanical energy was lost to air resistance? [48%]

57. A 7 kg bowling ball is lifted 2.2 m into a storage rack. Calculate the increase in the ball's gravitational potential energy. [ΔW_p=150.92 J]

58.A bobsled slides down an ice track starting (at zero initial speed) from the top of a 58.6 m high hill. The acceleration of gravity is 9.8 m/s^2. Neglect friction and air resistance and determine the bobsled's speed at the bottom of the hill. [$v = 33.9$ m/s]

59. A rope with tension T = 150 N pulls a 15-kg block 3 m up an incline ($\alpha = 25°$). The coefficient of kinetic friction is μ = 0.2. Find a) the acceleration of the block; b) the work done by each force acting on the block.



[$a = 4.09$ m/s²; A(tension) =450 J; A(gravity) = – 187 J, A(normal) = 0; A(friction) = – 80J]

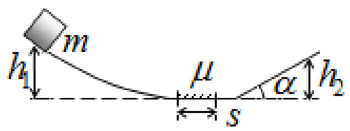
60. A 2 kg block slides along a frictionless horizontal with a speed of 6.2 m/s. After sliding a distance of 5 m, the block makes a smooth transition to a frictionless ramp inclined at an angle of 46° to the horizontal. The acceleration of gravity is 9.81 m/s². How far up the ramp does the block slide before coming momentarily to rest? [$s = 2.7\,\text{m}$]

61. A winch lifts a 150 kg crate 3 m upwards with acceleration of 0.5 m/s^2. How much work is done by the winch? How much work is done by gravity? [A(tension) = 4640 J; A(gravity) = – 4410 J]

62. What is the work done by friction in slowing a 10.5-kg block travelling at 5.85 m/s to a complete stop in a distance of 9.65 m? What is the kinetic coefficient of friction? [A(friction)= 179.67 J; $\mu = 0.18$]

63. An ideal spring with a spring constant of 500 N/m is firmly attached to a table in a horizontal orientation and is compressed 0.25 m from its natural length. What horizontal speed can it give to a 1 kg box when released? Ignore any dissipative effects and assume the surface is perfectly level. [$v = 5.59\,\text{m/s}$]

64. A 50-N force is applied horizontally to a 12-kg block which is initially at rest. After travelling 6.45 m, the speed of the block is 5.9 m/s. What is the coefficient of kinetic friction? [$\mu = 0.15$]



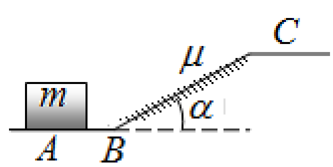
65. A 5-kg block slides from rest at a height of $h_1 =$ 1.75 m down to a horizontal surface where it passes over a 2 m rough patch. The rough patch has a coefficient of kinetic friction $\mu = 0.25$. What height, h_2, does the block reach on the $\alpha = 30°$ incline? [$h_2 = h_1 - \mu \cdot \Delta x = 1.25\,\text{m}$]

66. A 5-kg block slides from rest at a height of h_1 = 1.75 m down to a smooth horizontal surface until it encounters a rough incline. The incline has a

coefficient of kinetic friction μ= 0.25. What height, h_2, does the block reach on the $\alpha = 30°$ incline?

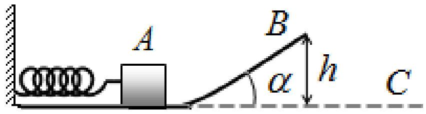
$$[h_2 = \frac{h_1}{1 + \left(\mu / \tan\alpha\right)} = 1.22\,\text{m}]$$

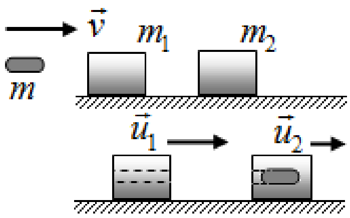
67. A block of mass 5 kg starts at point A with a speed of 15 m/s on a flat frictionless surface. At point B, it encounters an incline with coefficient of kinetic friction μ = 0.15. The block makes it up the incline to a second flat frictionless surface. What is the work done by friction? What is the velocity of the block at point C? The incline is 2.2 m long at an angle



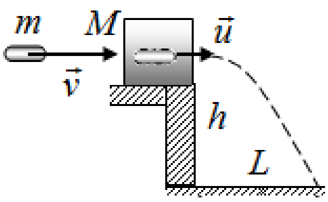
$\alpha = 15°$. [$A_{fr} = -15.635$ J; $v_C = \sqrt{2A_{fr} / \left(m - 2g \cdot \Delta x \cdot \sin\alpha + (v_A)^2\right)} = 14.4$ m/s]

68. At point A in the figure shown below, a spring (spring constant k = 1000 N/m) is compressed 50 cm by a 2 kg block. When released the block travels over the frictionless track until it is launched into the air at point B. It lands at point C. The inclined part of the track makes an angle of $\alpha = 55°$ with the horizontal and point B is a height h = 4.5 m above the ground. How far horizontally is point C from point B? [$\Delta x = 5.52$ m]



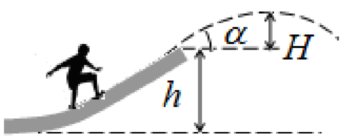


69. A 3.5-g bullet is fired horizontally at two blocks at rest on a frictionless table. The bullet passes through block of mass m_1 = 1.16 kg and embeds itself in block of mass m_2 = 1.85 kg. The blocks end up with speeds v_1 = 0.56 m/s and v_2 = 1.36 m/s. Neglecting the material removed from the first block by the bullet, find the speed of the bullet as it (a) enters and (b) leaves the first block [u_1 = 905.6 m/s; u_2 = 720m/s]



70. A 4.4 g bullet is fired into a block of wood with a mass of 21.3 g. The wood block is initially at rest on a 1.3 m tall post. After the collision, the wood block and bullet land 2.4 m from the base of the post. Find the initial speed of the bullet. [v = 27.2 m/s]

71. A 47-gram golf ball is driven from the tee with an initial speed of 52 m/s and rises to a height of 24.6 meters. a) Neglect air resistance and determine the kinetic energy of the ball at its highest point. b) What is its speed when it is 8 meters below its highest point? [52.2 kJ; 48.8 m/s]



72. A skateboarder moving at 5.4 m/s along a horizontal section of a track that is slanted upward by 48° above the horizontal at its end, which is $h = 0.4$ meters above the ground. When she leaves the track, she follows the characteristic path of projectile motion. Ignoring friction and air resistance, find the maximum height H to which she rises above the end of the track. [$H = 0.6$ m]

73. A crate is given an initial speed of 3.1 m/s up the 29°-plane. Assume coefficient of friction $\mu = 0.17$. How far up the plane will it go? How much time elapses before it returns to its starting point? [$s = 0.78$; $t = 0.5 + 0.69 = 1.19$ s]

74. A 3.3-kN piano is lifted by three workers at a constant speed to an apartment 28.8 m above the street using a pulley system fastened to the roof of the building. Each worker is able to deliver 197 W of power, and the pulley system is 75% efficient (so that 25% of the mechanical energy is lost due to friction in the pulley). Neglecting the mass of the pulley, find the time required to lift the piano from the street to the apartment. [t = 214; s =3.57 min]

75. A disk with a mass of 2 kg and a radius of 0.25 m is rolling without slipping with a constant angular velocity of 4 rad/s. How far does the center of mass move in 6 seconds? [$s = 6$ m]

76. A 20 kg square slab with side lengths 3 m rotates around an axis perpendicular to the slab. A particle of mass 0.6 kg sits on each of the corners of the square. Also, a particle of mass 0.8 kg sits at the center of each of the edges. What is the moment of inertia of the system? [I = 48 kg·m^2]

77. A thin hoop rolls without sliding along the floor. Find the ratio of its translational kinetic energy of the center of mass to its rotational kinetic energy about an axis through its center. [$W_t / W_r = 1$]

78. A uniform solid sphere of radius 0.1 m rolls smoothly across a horizontal table at a speed o.5 m/s with total kinetic energy 0.7 J. Find the mass of the sphere. [$m = 4$ kg]

79. A CD has a mass of 17 g and a radius of 6 cm. When inserted into a player, the CD starts from rest and accelerates to an angular velocity of 21 rad/s in 0.8 s. Assuming the CD is a uniform solid disk, determine the net torque acting on it. [$M = 8 \cdot 10^{-4}$ N·m]

80. A stone is suspended from the free end of a wire that is wrapped around the outer rim of a pulley. The pulley is a uniform disk with mass 10 kg and radius 50 cm and turns on frictionless bearings. You measure that the stone travels 12.6 m in the first 3 seconds, starting from rest. Find (a) the mass of the stone and (b) the tension in the wire. [$m = 2$ kg; $T = 14$ N]

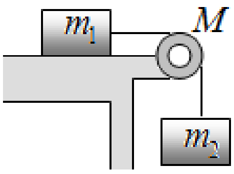
81. The moment of inertia of an oxygen molecule about an axis through the centre of mass and perpendicular to the line joining the atoms is $1.95 \cdot 10^{-46}$ kg-m^2. The mass of an oxygen atom is $2.66 \cdot 10^{-26}$ kg. What is the distance between the atoms? Treat the atoms as particles. [$l = 1.21 \cdot 10^{-10}$ m]

82. A disk with a mass of 2 kg and a radius of 0.25 m is rolling without slipping with a constant angular velocity of 4 rad/s. How far does the center of mass move in 6 seconds? [$s = 6$ m]

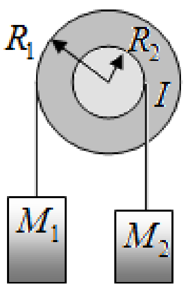
83. If a 32 N·m torque on a wheel causes angular acceleration of $\varepsilon = 25$ rad/s². What is the wheel's rotational inertia? [$I = 1.28$ kg·m²]

84. A 32 kg wheel, essentially a thin hoop with radius 1.2 m is rotating at 280 rev/min. It must be brought to a stop in 15 s. (a) How much work must be done to stop it? (b) What is the required power? [$A = -1.98 \cdot 10^4$ J; $P = 1.32$ kW]

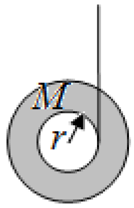
85. A wheel of radius 0.398 m is mounted on a frictionless horizontal axis. The rotational inertia of the wheel about the axis is 0.0576 kg·m2. A massless cord wrapped around the wheel is attached to a 0.79 kg block that slides on a horizontal frictionless surface. If a horizontal force of magnitude F = 3.52 N is applied to the block, what is the angular acceleration of the wheel? Take the clockwise direction to be the negative direction and assume the string does not slip on the wheel. [$\varepsilon = FR/\left(I + mR^2\right)$=7.67 rad/s²]



86. Two blocks (m_1 and m_2) are connected over a pulley: m_1 is on the table and m_2 is suspended from the pulley by the rope. The pulley has mass M and radius R. What is the acceleration of the blocks and the tension in the rope on either side of the pulley? [$a = m_2 g/\left(m_1 + m_2 + 0.5M\right)$;
$T_1 = \left(m_1 m_2 g\right)/\left(m_1 + m_2 + 0.5M\right)$; $T_2 = m_2 g\left(m_1 + 0.5M\right)/\left(m_1 + m_2 + 0.5M\right)$]



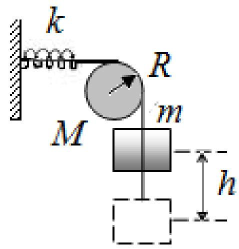
87. A pulley has a moment of inertia of $I = 10$ kg-m². Two masses M_1= 4 kg and M_2 = 2 kg are attached to strings which are wrapped around different parts of the winch which have radii R_1 = 40 cm and R_2 = 25 cm. a) How are the accelerations (linear and angular) of the two masses and the pulley related? b) Determine the angular acceleration of the masses. Recall that each object needs a separate free body diagram. c) What are the tensions in the strings?
[$\varepsilon = g\left(R_1 M_1 - R_2 M_2\right)/\left(I + M_1 R_1^2 + M_2 R_2^2\right) = 1.002$ rad/s²; $a_1 = \varepsilon R_1 = 0.4008$ m/s²; $a_2 = \varepsilon R_2 = 0.06$ m/s²; $T_1 = 38.6$ N; $T_2 = 20.12$ N]



88. A yo-yo has a mass M, a moment of inertia I, and an inner radius r. A string is wrapped around the inner cylinder of the yo-yo. A person ties the string to his finger and releases the yo-yo. As it falls, it does not slip on the string (i. e. it rolls). Find the acceleration of the yo-yo. [$a = mg/\left(m + I/r^2\right)$]

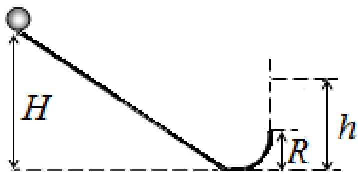
89. A yo-yo is made by wrapping the long string around a 500-g uniform disk. The string is fastened to the ceiling, and the yo-yo is released. What is the tension in the string and the magnitude of the acceleration of the yo-yo as it falls? [$T = mg/3 = 1.63\,\mathrm{N}$, $a = 2g/3 = 6.53\,\mathrm{m/s^2}$]

90. A block of mass m is connected by a string of negligible mass to a spring with spring constant k which is in turn fixed to a wall. The spring is horizontal and the string is hung over a pulley such that the mass hangs vertically. The pulley is a solid disk of mass M and radius R. As shown in the figure, the spring is initially in its equilibrium position and the system is not moving. Use energy methods, to determine the speed v of the block after it has fallen a distance h. [$v = \sqrt{\left(2mg - kh^2\right)/\left(m + 0.5M\right)}$]



91. A solid ball and a hollow ball, each with a mass of m and radius of r, start from rest and roll down a ramp of length l = 2 m at an incline of 34°. An ice cube of the same mass slides without friction down the same ramp. What are the speeds of the solid ball, hollow ball and the cube at the bottom of the incline?
[$v_1 = \sqrt{1.43 gl \sin\alpha}$ =3.96 m/s; $v_2 = \sqrt{1.2 gl \sin\alpha} = 3.63$ m/s;
$v_3 = \sqrt{2gl \sin\alpha} = 4.68$ m/s]

92. A small circular object with mass m and radius r has a moment of inertia given by $I = cmr^2$. The object rolls without slipping along the track shown in the figure. The track ends with a ramp of height R = 2 m that launches the object vertically. The object starts from a height H = 8.5 m. To what maximum height will it rise after leaving the ramp if c = 0.37? [$h = 4.74$ m]



93. A sphere of mass 140 kg rolls so that its center of mass has speed of 0.15 m/s. How much work must be done on the sphere to stop it. [$A = -2.21\,\mathrm{J}$]

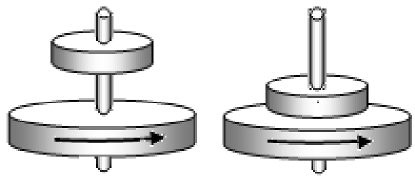
94. A ring is given an initial speed of 7 m/s at its center of mass. Then it rolls smoothly up the incline. What is the speed of the center of mass of the ring at a height of 5 m? [$v=\sqrt{2\left(v_0^2-gh\right)}=0$]

95. A child pushes a merry-go-round with a tangential force of 55 N. If the child pushes it through exactly one full circle, and the merry-go-round has a radius of 1.15 m, how much work does she do? [518.4 J]

96. A door with width l = 1 m and mass M = 15 kg is hinged on one side so that it can rotate freely. A bullet is fired into the exact centre of the door. The bullet has mass m=25 g and a speed of 400 m/s. What is the angular velocity of the door with respect to the hinge just after the bullets embeds itself in the door? The door may be treated as a thin rectangular sheet. The bullet may be treated as a point mass. [ω =1 rad/s]

97. A potter's wheel, with rotational inertia 6.4 kg·m², is spinning freely at 19 rpm. The potter drops a 2.7-kg lump of clay onto the wheel, where it sticks a distance of 46 cm from the rotation axis. What is the subsequent angular speed of the wheel? [$n=17.4$ rev/min]

98. The lower disk of mass m_1 = 440 g and radius R_1 = 3.5 cm, is rotating at n_1 =180 rpm on a frictionless shaft of negligible radius. The upper disk, of mass m_2 = 270 g and radius R_2 = 2.3 cm, is initially not rotating. It drops freely down the shaft onto the lower disk, and frictional forces act to bring the two disks to a common rotational speed. (a) What is that speed? (b) What fraction of the initial kinetic energy is lost to friction? [$n_2=142$ rpm; $\left(W_{k1}-W_{k2}\right)/W_{k1}=0.209$]



99. The lower disk (see previous problem) is initially spinning about a frictionless shaft while the upper disk is stationary. The upper disk drops onto the lower one, and they come to a common angular speed. If one-third of the initial energy is lost in the process, how do the moments of inertia of the two disks compare? [$I_2=I_1/2$]

100. A skater's body has rotational inertia 4.2 kg·m² with his fists held to his chest and 5.7 kg·m² with arms outstretched. The skater is twirling at 3 rev/s while holding 2.5-kg weights in each outstretched hand; the weights are 76 cm from his rotation axis. If he pulls his hands in to his chest, how fast will he be twirling? (Assume that the weights are essentially pulled into the axis of rotation) [$n_2=6.13$ rev/s]

Chapter 3. OSCILLATIONS AND WAVES

3.1. OSCILLATIONS

Oscillations are periodic changes of any quantity. *Mechanical oscillations* (*vibrations*) are the motions that repeat themselves over and over again. Oscillations are the periodic motions of the material point or material object.

The *types of oscillations*:

1. ***Free*** (*characteristic, natural*) oscillations take place in systems which begin to move after a kick or after upsetting a balance. There are ***undamped*** (*continuous*, *persistent*) oscillations and ***damped*** (*convergent, decaying*) oscillations which take place in systems without or with energy loss, correspondingly.

2. ***Forced*** (*constrained*) oscillations occur in systems exposed to action of external periodic force.

3.1.1. Free undamped oscillations. Simple harmonic motion and its characteristics

The physical system making oscillations about an equilibrium position is an *oscillator*.

The simple type of oscillations occurring under the law of cosine or sine is ***simple harmonic motion*** (SHM), and the system in this case is called a ***simple harmonic oscillator*** (SHO).

The ***equation of free undamped oscillations*** or ***simple harmonic motion*** is

$$x = A\cos(\omega_0 t + \alpha), \tag{3.1}$$

where x is the distance travelled by vibrating particle at any instant of time t from its mean position that is called ***displacement***; A is the ***amplitude*** that is defined as the maximum displacement of the particle from mean position (fig. 3.1, *a*).

The time varying quantity $(\omega_0 t + \alpha)$ in (3.1) is a ***phase*** of oscillations, and the constant α is called the ***initial phase*** *angle* (*phase constant, the phase angle*). The value of α depends on the displacement and velocity of the particle at $t = 0$. The quantity ω_0 is a ***natural*** (*own*) ***angular frequency*** of the motion.

The time required for one complete vibration (complete to-and-fro movement) is called the ***period*** (T). The ***frequency*** (ν) is a quantity that is equal to the number of vibrations per second.

During one period the phase of oscillation changes by 2π radians, thus,

$$\omega_0\left(t+T\right)+\alpha=\left(\omega_0 t+\alpha\right)+2\pi\,. \tag{3.2}$$

Therefore,

$$\omega_0 T=2\pi\,,$$

$$\omega_0=\frac{2\pi}{T}=2\pi\upsilon_0\,. \tag{3.3}$$

$$\left[T\right]=\text{s},\quad \left[\omega_0\right]=\text{rad/s},\quad \left[\upsilon_0\right]=\text{s}^{-1}=\text{Hertz}=\text{Hz}.$$

To find the ***velocity*** $v\left(t\right)$ as a function of time (fig. 3.1, *b*) the derivative of the position function $x\left(t\right)$ in (3.1):

$$v=\frac{dx}{dt}=\dot{x}=-A\omega_0\sin(\omega_0 t+\alpha)=-v_{\mathrm{m}}\sin(\omega_0 t+\alpha)\,, \tag{3.4}$$

The *velocity amplitude* (the maximum variation of the velocity) is

$$v_m=A\omega_0\,. \tag{3.5}$$

Differentiating of the velocity function (3.4) gives an ***acceleration*** function $a\left(t\right)$ (fig. 3.1, *c*) of the particle

$$a=\frac{dv}{dt}=\frac{d^2x}{dt^2}=\ddot{x}=-A\omega_0^2\cos(\omega_0 t+\alpha)=-a_{\mathrm{m}}\cos(\omega_0 t+\alpha)=-\omega_0^2 x. \tag{3.6}$$

The *acceleration amplitude* is

$$a_m=A\omega_0^2\,. \tag{3.7}$$

The equation for acceleration (3.6) can be applied for the description of the ***force*** responsible for SHM.

$$F=ma=m\ddot{x}=-mA\omega_0^2\cos(\omega_0 t+\alpha)=-F_{\mathrm{m}}\cos(\omega_0 t+\alpha)=-m\omega_0^2 x=-kx\,. \tag{3.8}$$

The minus sign means that the direction of the force on the particle is *opposite* the direction of the displacement of the particle. That is, in SHM the force behaves similar to the restoring (elastic) force in the sense that it fights against the displacement, attempting to restore the particle to the center point at $x = 0$. In other words, this force is proportional to the displacement $F = -kx$. The force that is not elastic in their nature, but subject to the Hook's law, is called *quasi-elastic force*.

This equation is the same as the equation (2.12) for Hook's law, but the coefficient k is not the spring constant. It is the coefficient of quasi-elastic force and depends on the characteristics on the vibrating particle

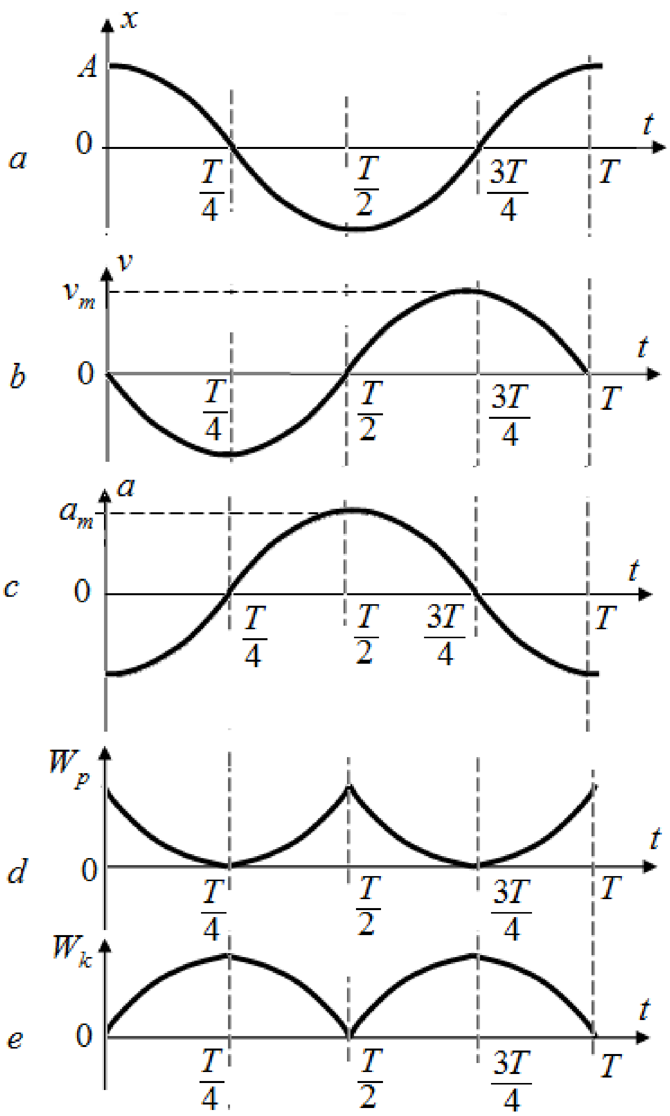


Figure 3.1

$$k = m\omega_0^2. \tag{3.9}$$

Two forms of energy are involved during vibrations. In standard example of a the block-spring system oscillating according to the equation (3.1), they are the ***potential energy*** (fig. 3.1, d) stored in the spring

$$W_p = \frac{kx^2}{2} = \frac{m\omega_0^2 A^2}{2}\cos^2(\omega_0 t + \alpha) = (W_p)_m \cos^2(\omega_0 t + \alpha), \tag{3.10}$$

and the ***kinetic energy*** (fig. 3.1, e) of moving block

$$W_k = \frac{mv^2}{2} = \frac{m\omega_0^2 A^2}{2}\sin^2(\omega_0 t + \alpha) = (W_k)_m \sin^2(\omega_0 t + \alpha). \tag{3.11}$$

The maximum magnitudes of potential $\left(W_p\right)_m$ and kinetic $\left(W_k\right)_m$ energies are equal to each other and equal to the ***total energy*** of a simple harmonic oscillator

$$W = W_p + W_k = \frac{m\omega_0^2 A^2}{2}\cos^2\left(\omega_0 t + \alpha\right) + \frac{m\omega_0^2 A^2}{2}\sin^2\left(\omega_0 t + \alpha\right) = \frac{m\omega_0^2 A^2}{2}. \quad (3.12)$$

The total mechanical energy of a simple harmonic oscillator is a constant of the motion and is proportional to the square of the amplitude A^2.

Energy is continuously being transformed between potential energy stored in the spring and kinetic energy of the block. Figure (Fig. 3.1) illustrates the dependencies of a) position, b) velocity, c) acceleration, d) potential energy, and e) kinetic energy of the block-spring system for one full period of the motion.

3.1.2. Pendulums

1. A ***spring pendulum*** **(***or linear simple harmonic oscillator***)** is a system consisting of a mass m on a spring which mass can be neglected in comparison with a mass m (fig. 3.2). The equation of motion of this system according to Newton's 2nd law is

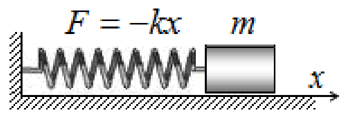


Figure 3.2

$$m\ddot{x} = -kx, \quad (3.13)$$

$$\ddot{x} + \omega_0^2 x = 0. \quad (3.14)$$

The solution of this differential equation is

$$x = A\cos(\omega_0 t + \alpha),$$

where the angular frequency is

$$\omega_0^2 = \frac{k}{m}, \quad (3.15)$$

and period of the motion is

$$T = 2\pi\sqrt{\frac{m}{k}}. \quad (3.16)$$

2. A ***simple pendulum*** is a particle-like bob of mass m suspended by a light string of length l that is fixed at the upper end (fig. 3.3). The motion occurs in the vertical plane along the arc of a circle and is driven by gravitational force. The equation of motion of this system according to the Newton's 2nd law for rotation is

$$I\varepsilon = M,$$

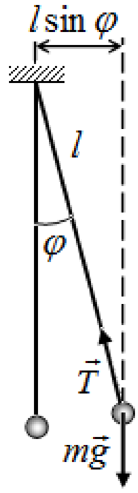


Figure 3.3

Taking into account that the rotational inertia of the point mass is $I = mr^2$, the torque of the gravitational force is $mgl\sin\varphi$, the equation of the motion will be

$$ml^2\ddot{\varphi} = -mgl \cdot \sin\varphi. \tag{3.17}$$

The negative sign in (3.17) indicates that the net tangential force (the vector sum of $m\vec{g}$ and $\vec{T}$) acts towards the equilibrium position. If we assume that φ is small, we can use the approximation $\sin\varphi \approx \varphi$, and the equation of motion for the simple pendulum becomes

$$\ddot{\varphi} + \frac{g}{l}\varphi = 0.$$

$$\ddot{\varphi} + \omega_0^2\varphi = 0.$$

Now we have an expression that has the same form as (3.14), and we conclude that the motion for small amplitudes of oscillation is simple harmonic motion. Therefore, the function $\varphi = \varphi(t)$ can be written as

$$\varphi = \varphi_{\mathrm{m}}\cos(\omega_0 t + \alpha), \tag{3.18}$$

where φ_{m} is the maximum angular position. The angular frequency is

$$\omega_0 = \sqrt{\frac{g}{l}} \tag{3.19}$$

and the period of oscillations is

$$T = 2\pi\sqrt{\frac{l}{g}}. \tag{3.20}$$

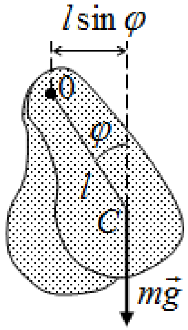


Figure 3.4

3. A ***physical* (*compound*) *pendulum*** is a rigid body oscillating about a fixed axis passing through the *point of suspension* located above its center of mass. This object cannot be approximated as a point mass.

Consider the object pivoted at a point O that is at distance l from its center of mass (fig. 3.4). The gravitational force $m\vec{g}$ provides a torque about the axis passing through the point O and the magnitude of that torque is $M = mgx \cdot \sin\varphi$. Using the rotational form of Newton's 2nd law (2.103), we obtain

$$I\ddot{\varphi} = -mgx \cdot \sin\varphi, \tag{3.21}$$

where I is the moment of inertia of the object about the axis through O and the negative sign indicates that the torque about O tends to decrease the angle φ.

Assuming the angle φ is small and using the approximation $\sin\varphi \approx \varphi$, we reduce (3.21) to the equation

$$\ddot{\varphi} + \frac{mgx}{I}\varphi = 0. \tag{3.22}$$

Putting $\omega_0^2 = \dfrac{mgx}{I}$, solve it for φ:

$$\varphi = \varphi_{\mathrm{m}} \cos(\omega_0 t + \alpha), \tag{3.23}$$

where φ_{m} is the *maximum angular position,* and the *angular frequency* is

$$\omega_0 = \sqrt{\frac{mgx}{I}}\ . \tag{3.24}$$

The period is

$$T = 2\pi\sqrt{\frac{I}{mgx}} = \sqrt{\frac{L}{g}}\,. \tag{3.25}$$

where $L = \dfrac{I}{mx}$ is the *equivalent length* or the length of an equivalent simple pendulum, i. e., if we want to construct a simple pendulum having the time period as the given compound pendulum, the length of the simple pendulum should be L.

3.1.3. Relation between linear SHM and uniform circular motion. Superposition of unidirectional oscillations

The expression for SHM $x = A\cos(\omega_0 t + \alpha)$ has a very simple interpretation in terms of relations between SHM along a line and a uniform motion in a circle.

Consider a particle moving on a circle of radius A with uniform speed. Let T be the period of the circular motion. At time $t = 0$ the particle is at point 1 (fig. 3.5), after the time interval t, it is at point 2. Let point B be the foot of the perpendicular drawn from point 2 on the diameter (A, –A) of the circle. As the particle moves around the circle, the point B moves from point A to point –A, and back to A as the particle moves to point 1. During time t when the particle moves to point 2 along the circle, the radius sweeps an angle φ. The angular velocity of the circular motion is

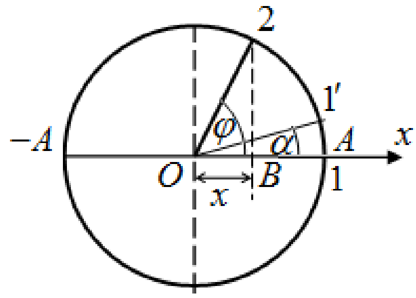


Figure 3.5

$$\omega_0 = \frac{\text{angle swept}}{\text{time taken}} = \frac{\varphi}{t} = \frac{2\pi}{T}, \tag{3.26}$$

where $t = T$, $\varphi = 2\pi$, or $\varphi = \omega_0 t$.

Since $OB = A\cos\varphi = A\cos\omega_0 t$, $x = A\cos\omega_0 t$. Thus SHM can be described as the projection of a uniform circular motion on the diameter of the circle. The orbiting point (in general, the body) is called the *reference body* and the circle along which it moves id called the *reference circle*. The magnitude ω_0 of a SHM is the same as the angular speed of the reference body.

If the moving point starts from other position then the point 1, we'll get

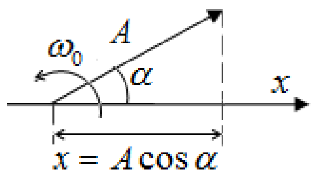


Figure 3.6

$$x = A\cos(\omega_0 t + \alpha).$$

Hence the phase constant α is the measure of how far the point is from position 1, or the point B is from the center of the reference circle.

Now we can visualize the SHM using the rotating vector representation. If vector A rotates at an angular speed ω_0 in the counter-clockwise sense, the projection of this vector vibrates in limits $(+A, -A)$ according to the law $x = A\cos(\omega_0 t + \alpha)$, i. e. the projection oscillates harmonically.

Thus, the simple harmonic motion can be represented as a vector whose magnitude is equal to the amplitude of oscillations and the direction forms an angle equal to the initial phase of oscillation with an x-axis.

Let's consider ***two collinear*** (*unidirectional*) ***harmonic oscillations***, each of frequency ω_0 but different amplitudes and initial phases.

$$\begin{cases} x_1 = A_1 \cos\left(\omega_0 t + \alpha_1\right), \\ x_2 = A_2 \cos\left(\omega_0 t + \alpha_2\right). \end{cases} \qquad (3.27)$$

The resultant oscillation is obtained by their superposition

$$x = x_1 + x_2 = A_1 \cos\left(\omega_0 t + \alpha_1\right) + A_2 \cos\left(\omega_0 t + \alpha_2\right) = A \cos\left(\omega_0 t + \alpha\right). \qquad (3.28)$$

As a result of their superposition, we obtain the resulting simple harmonic motion which ***amplitude*** A may be compute according to the cosine law

$$A^2 = A_1^2 + A_2^2 - 2A_1A_2 \cos\left[\pi - \left(\alpha_2 - \alpha_1\right)\right] = A_1^2 + A_2^2 + 2A_1A_2 \cos\left(\alpha_2 - \alpha_1\right),$$

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2 \cos\Delta\alpha}\,, \qquad (3.29)$$

and an ***initial phase*** α is

$$\tan\alpha = \frac{A_1 \sin\alpha_1 + A_2 \sin\alpha_2}{A_1 \cos\alpha_1 + A_2 \cos\alpha_2}. \qquad (3.30)$$

Figure 3.7 demonstrates the vector method of finding amplitude A and initial phase α of resultant motion.

Special cases depend on the initial phase difference $\Delta\alpha = \alpha_2 - \alpha_1$.

1. If $\Delta\alpha = \pm 2\pi k$, where $k = 0,1,2,...$, $\cos\Delta\alpha = 1$. The amplitude of the resulting oscillations is maximum

$$A = \left|A_1 + A_2\right|. \qquad (3.31)$$

2. If $\Delta\alpha = \pm\left(2k+1\right)\pi$, where $k = 0,1,2,...$, $\cos\Delta\alpha = -1$. The resultant amplitude is minimum

$$A = \left|A_1 - A_2\right|. \qquad (3.32)$$

3. $\Delta\alpha = \pm(2k+1)\frac{\pi}{2}$, where $k = 0,1,2,...$, $\cos\Delta\alpha = 0$. The resultant amplitude is

$$A = \sqrt{A_1^2 + A_2^2}\,. \tag{3.33}$$

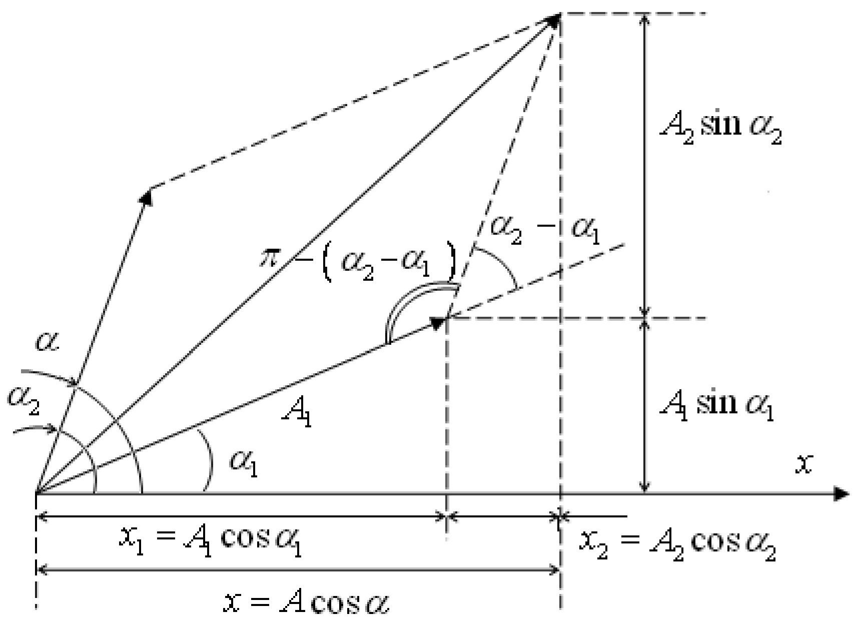


Figure 3.7

If the adding oscillations have different frequencies and the phase difference between vibrations is continually changing, the resultant oscillation is complicated function of time. But if two SHMs are close in frequency

$$\begin{cases} x_1 = A_1\cos\omega_1 t, \\ x_2 = A_2\cos\omega_2 t, \end{cases} \tag{3.34}$$

the combined disturbance exhibits ***beats***

$$x = 2A\cos\left(\frac{\omega_1 - \omega_2}{2}t\right)\cos\left(\frac{\omega_1 + \omega_2}{2}t\right), \tag{3.35}$$

where the first cosine term is slowly oscillating and the second is fast oscillating.

The condition for beats is

$$|\omega_1 - \omega_2| \ll \omega_1 + \omega_2,$$

the combines displacement can be fit within an envelope defined by the pair of equations

$$x = \pm 2A\cos\left(\frac{\omega_1 - \omega_2}{2}t\right), \tag{3.35}$$

which describes a relatively slow amplitude modulation of the combined oscillations.

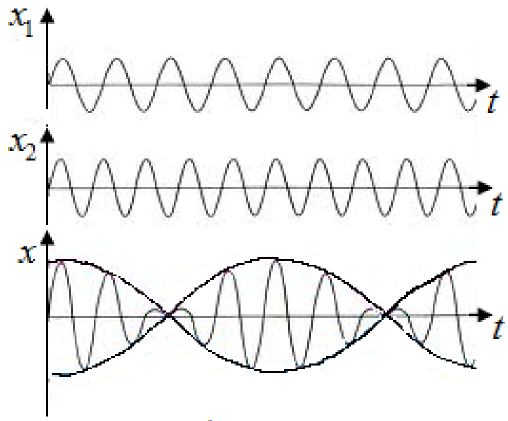


Figure 3.8

The beat frequency is equal to the absolute value of the difference in frequencies

$$\omega_{\text{beat}} = \left|\omega_1 - \omega_2\right|.$$

The phenomenon of beats is of great practical importance. For example, beats can be used to found the small phase difference between two sources of sound in tuning the musical instruments.

3.1.4. Superposition of oscillations at right angle

Two simple harmonic motions are

$$\begin{cases} x = A\cos\omega_0 t, \\ y = B\cos(\omega_0 t + \alpha), \end{cases} \tag{3.37}$$

where x and y are the displacements along mutually perpendicular directions.

Since the initial phase of the oscillation along x-axis is zero and the phase of oscillation along y-axis is α, the initial phase difference is α.

The resultant motion of the vibrating object is along elliptical path since the system of equations (3.37) is equations of ellipse in parametrical form. Analytical equation can be found by eliminating the parameter t from the equations (3.37).

From the first equation of the system, $\cos\omega_0 t = \frac{x}{A}$, hence,

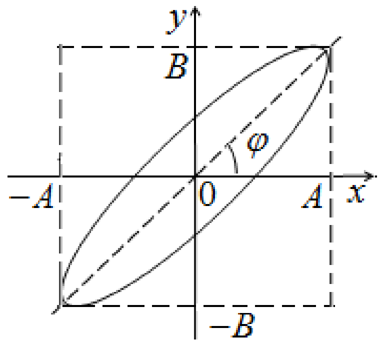


Figure 3.9

$$\sin\omega_0 = \pm\sqrt{1-\frac{x^2}{A^2}}\,. \tag{3.38}$$

Substitution of (3.38) into the second equation gives

$$\frac{y}{B} = \cos(\omega_0 t+\alpha) = \cos\omega_0 t\cdot\cos\alpha - \sin\omega_0 t\cdot\sin\alpha = \frac{x}{A}\cos\alpha \mp \sqrt{1-\frac{x^2}{A^2}}\cdot\sin\alpha\,.$$

On rearranging terms and squaring both sides, we obtain

$$\frac{y}{B} - \frac{x}{A}\cos\alpha = \mp\sqrt{1-\frac{x^2}{A^2}}\cdot\sin\alpha\,.$$

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} - \frac{2xy}{AB}\cos\alpha = \sin^2\alpha\,. \tag{3.39}$$

Equation (3.39) represents an *ellipse* inscribed within a rectangle of sides $2A$ and $2B$. The major axis of ellipse is inclined at an angle φ with the x-axis such that

$$\tan 2\varphi = \frac{2AB\cos\alpha}{A^2-B^2}\,. \tag{3.40}$$

Specializing to particular values of initial phase difference α we distinguish *several cases*.

1. $\alpha = \pm 2\pi k\,, k = 0,1,2,...$

$$y = \frac{B}{A}x\,. \tag{3.41}$$

The motion with a frequency ω_0 and an amplitude $\sqrt{A^2+B^2}$ (fig. 3.10, *a*) is *rectilinear* and takes place along the diagonal of the rectangle such that x and y

always have the same sign. A straight line passing through the origin lies in the first and the third quarters of Cartesian coordinate system

2. $\alpha = \pm(2k+1)\pi$, $k = 0,1,2,\dots$.

The motion with an angular frequency ω_0 and an amplitude $\sqrt{A^2+B^2}$ (fig. 3.10, *b*) is *rectilinear* and takes place along the diagonal of the rectangle such that x and y always have the opposite signs. A straight line passing through the origin lies in the second and the forth quarters of Cartesian coordinate system.

$$y = -\frac{B}{A}x. \tag{3.42}$$

3. $\alpha = \pm(2k+1)\frac{\pi}{2}$, $k = 0,1,2,\dots$.

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1. \tag{3.43}$$

The motion with an angular frequency ω_0 is along the *ellipse* (fig. 3.10, *c*) whose principal axes lie along x and y axes of Cartesian coordinate system (ellipse reduced to principal axes).

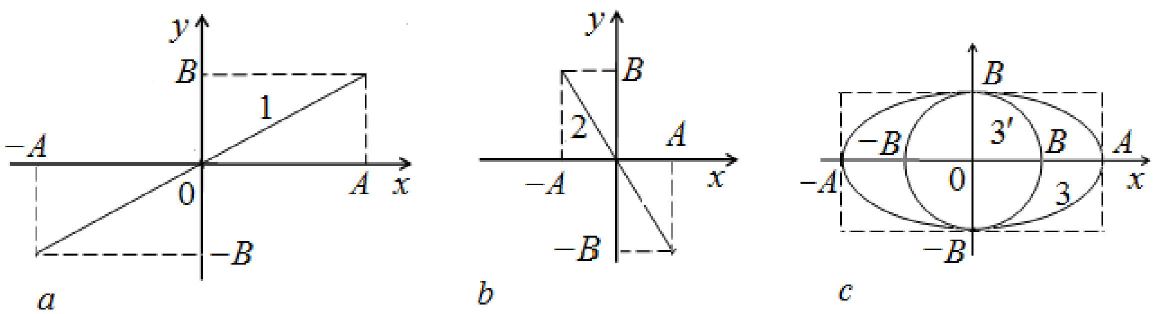


Figure 3.10

If semi-axes $A = B$, the equation of ellipse (3.43) reduces to

$$x^2 + y^2 = A^2, \tag{3.44}$$

which represents a circle ($3'$ in fig. 3.10, *c*) of radius A with center at the origin. This corresponds to uniform circular motion with angular speed ω_0.

3.1.5. Free damped oscillations

Observation of free oscillations of a real physical system reveals that the energy of the oscillator gradually decreases with time and the oscillator comes to rest. The energy has been losing off a vibrating system for various reasons, for example, such as conversion to heat via friction. The presence of friction to motion implies that frictional or damping force acts on the system. As a result, the amplitude of oscillations gradually diminishes with time. The reduction in amplitude (or energy) of an oscillator is called ***damping*** and the oscillations are said to be ***damped***.

Let's consider the object of mass m executing SHM in the medium where the damping force $\vec{F}_d$ acts on it. The equation of the motion is

$$m\vec{a} = \vec{F} + \vec{F}_d, \tag{3.45}$$

where $\vec{F}$ is restoring force. It is proportional to displacement $F = -kx$, where k is the coefficient of the restoring force (in most cases it is the spring constant).

For small velocities of motion the damping force

$$F_d = -rv, \tag{3.46}$$

where r is the coefficient of the damping force (in general, the coefficient of friction).

$[r]$ =kg·s^{-1}.

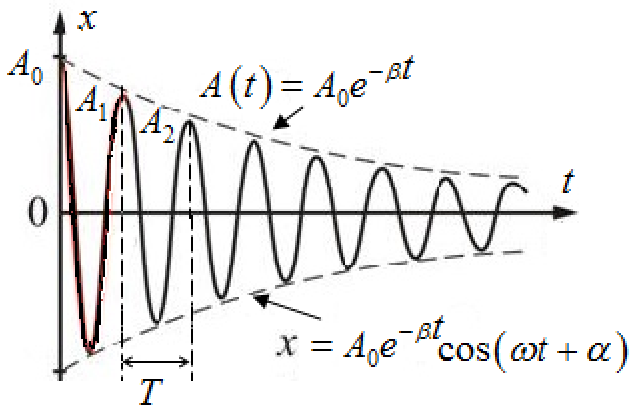


Figure 3.11

For one-dimensional case the equation of the motion according to Newton's 2nd law is

$$m\ddot{x} = -kx - r\dot{x}. \tag{3.47}$$

Equation (3.47) can be rewritten as

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0, \tag{3.48}$$

where

$$\beta = \frac{r}{2m} \tag{3.49}$$

is the ***damping coefficient***, and $\omega_0 = \sqrt{\frac{k}{m}}$ is the natural frequency, i. e., the frequency of the system without friction.

$[\beta] = \mathrm{s}^{-1}$.

The equation of the damped oscillator for small displacements and speeds (3.48) may be solved in the form which is the ***equation of free damped oscillations***.

$$x = A_0 e^{-\beta t}\cos(\omega t + \alpha). \tag{3.50}$$

The ***amplitude of damped oscillations*** varies under the exponential law

$$A(t) = A_0 e^{-\beta t}. \tag{3.51}$$

An ***angular frequency of damped oscillations*** is

$$\omega = \sqrt{\omega_0^2 - \beta^2}. \tag{3.52}$$

Due to weak damping condition, the period of damped oscillations is kept constant during oscillation process and thus is called the ***conditional period*** of damped oscillations

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\omega_0^2 - \beta^2}}. \tag{3.53}$$

A ***damping decrement*** is the quantity equaled to the ratio of the amplitudes corresponding to the instants of time distinguished by the period, i. e.,

$$D = \frac{A(t)}{A(t+T)} = \frac{A_0 e^{-\beta t}}{A_0 e^{-\beta(t+T)}} = e^{\beta T}. \tag{3.54}$$

A ***logarithmic damping decrement*** (*damping factor*)

$$\delta = \ln D = \beta T. \tag{3.55}$$

A ***quality factor***

$$Q = \frac{\pi}{\delta}. \qquad (3.56)$$

A quality factor is defined as the number of cycles required for the energy to fall off by factor of 535. (The origin of this numerical factor is $e^{2\pi}$, where $e = 2{,}71828\ldots$ is the base of natural logarithms). The terminology arises from the fact that friction is often considered a bad thing, so a mechanical device that can vibrate for many oscillations before it loses a significant fraction of its energy would be considered a high-quality device).

A ***relaxation time*** τ is the period of time required for the amplitude to fall off by factor e.

$$\tau = \frac{1}{\beta}. \qquad (3.57)$$

3.1.6. Forced oscillations

While damping measures the energy loss from a vibrating system, it is also possible to put energy into a vibrating system. This is called *driving* or *forcing*. If the damped oscillator is driven by external periodic force $F_0 \cos\omega t$, the equation of motion is

$$m\ddot{x} = -kx - r\dot{x} + F_0 \cos\omega t. \qquad (3.58)$$

Putting $\beta = \frac{r}{2m}$ and $\omega_0 = \sqrt{\frac{k}{m}}$ similar to (3.47), we obtain the ***equation of forced oscillations***

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \left(\frac{F_0}{m}\right)\cos\omega t. \qquad (3.59)$$

The general solution of (3.59) consists of the general solution of the homogeneous equation (3.48) and any particular integral of (3.59):

$$x = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}\cos(\omega t - \arctan\frac{2\beta\omega}{\omega_0^2 - \omega^2}),$$

where ω is the ***frequency of a driving force***.

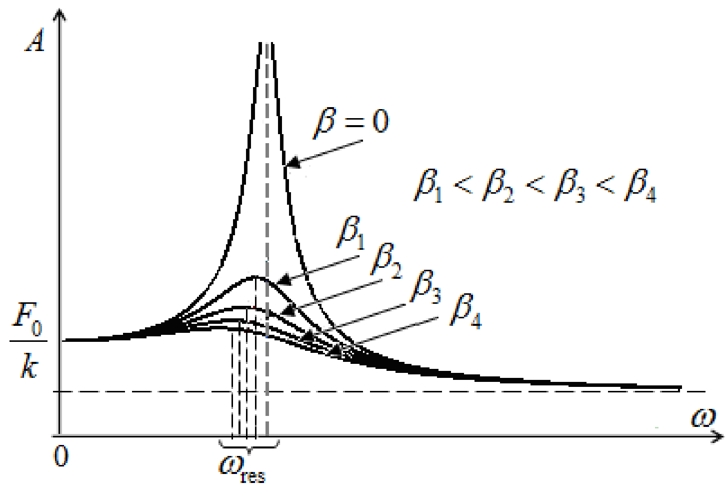


Figure 3.12

Resonance is the tendency of a vibrating system to respond most strongly to a driving force whose frequency is close to its own (natural) frequency of vibration (fig. 3.12). In an undamped system, the amplitude will be infinite.

For calculation of a resonance frequency let us find the maximum of function $A(\omega)$ by differentiation of a function

$$A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}, \tag{3.60}$$

$$\frac{dA}{d\omega} = \frac{F_0}{m}\left(-\frac{1}{2}\right)\left[\left(\omega_0^2 - \omega^2\right)^2 + 4\beta^2\omega^2\right]^{-\frac{3}{2}}\left[2\left(\omega_0^2 - \omega^2\right)(-2)\omega + 8\beta^2\omega\right] = 0,$$

$$4\omega\left(-\omega_0^2 + \omega^2 + 2\beta^2\right) = 0. \tag{3.61}$$

The cubic equation (3.61) has three roots:

1) $\omega = 0$ gives the minimal amplitude;

2) $\omega = -\sqrt{\omega_0^2 - 2\beta^2}$ is the physical nonsense;

3) the ***resonance frequency***

$$\omega_{\text{res}} = \sqrt{\omega_0^2 - 2\beta^2}\ . \tag{3.62}$$

Substitution of (3.62) into (3.60) gives the ***resonance amplitude***

$$A_{\text{res}} = \frac{F_0/m}{2\beta\sqrt{\omega_0^2 - \beta^2}}. \tag{3.63}$$

3.2. WAVES

3.2.1. Classifications of wave motion

Wave motion in a medium is a collective phenomenon that involves local interactions among the particles of the medium.

Waves are characterized by: 1) the disturbance in space and time; 2) the transfer of energy from one place to another; 3) the absence of substance transfer. A mechanical wave is a disturbance in the equilibrium positions of matter, the magnitude of which is dependent on location and on time. Waves transfer energy, momentum and information, but not mass.

Oscillations having arisen in one place of an elastic medium are transmitted to the next particles due to interaction and propagates with the velocity $\vec{v}$. Therefore, a ***wave*** is a disturbance that propagates through the medium. The line indicating a direction of propagation of wave is a *beam*.

Classification of the waves can be made using their different characteristics.

We can classify waves by medium where they spread: *mechanical* waves (the matter is medium), *electromagnetic* waves (electric and magnetic fields are the media), etc.

In accordance with orientation, the waves are divisible into transverse and longitudinal waves.

1. ***Transverse waves****:* displacements are *perpendicular* to the direction of propagation (fig. 3.13). For example, all electromagnetic waves (including light) are transverse.

Crest: a point of maximal displacement in the positive direction (upward displacement). *Trough*: a point of maximal displacement in the negative direction. Transverse mechanical waves are spread only in *solids*.

2. ***Longitudinal waves***: displacements are *parallel* to the direction of propagation (fig. 3.14). The example of longitudinal wave is sound.

Compression is a region where the medium is under compression. *Rarefaction* is a region where the medium is under tension. Longitudinal waves are spread in *solids*, *liquids* and *gases*.

If the source or origin of the wave oscillates at a frequency υ, then each point in the medium concerned oscillates at the same frequency. The geometrical place of points which the wave has reached up to a certain time is a *wave-front*. It is unique for the given wave process. The geometrical place of the points that are in the same phase is a *wave surface*. There are a lot of such surfaces.

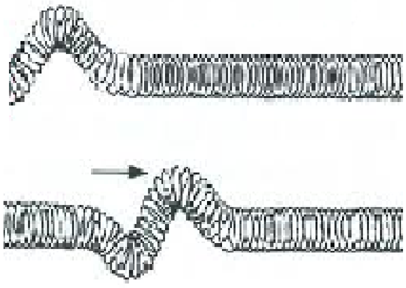


Figure 3.13

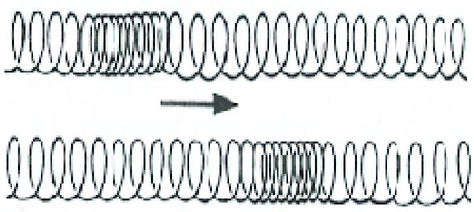


Figure 3.14

Waves spread out in all directions from every point on the disturbance that created them. If the disturbance is small, we may consider it as a single point. Then depending on the form of medium where the waves are spread, they may be *one-dimensional* (*linear*), *two-dimensional* (*circular*) and *three-dimensional* (*spherical*) waves.

Classifying waves by duration we have to note *episodic* (or *pulse*) waves when disturbance is momentary and sudden and *periodic* (or harmonic) waves when the disturbance repeats at regular intervals.

Classifying waves by propagation we distinguish travelling (progressive) and standing waves. *Travelling* waves are the waves that propagate in medium. *Standing* waves don't go anywhere, but they have regions where the disturbance of the wave is quite small, almost zero. These locations are called *nodes*. There are also regions where the disturbance is quite intense, greater than anywhere else in the medium, called *antinodes*.

3.2.2. Mathematical description of travelling wave

Waves propagate at a finite speed v (the ***wave speed***) that depends upon the type of wave, the composition and the state of the medium. The wave profile moves along at speed of wave v. If a snapshot is taken of a travelling wave, it is seen that it repeats at equal distances. The repeat distance is the *wavelength* λ. ***Wavelength*** is the distance between any point of a periodic wave and the next point corresponding to the same portion of the wave measured along the path of propagation. Wavelength is measured between adjacent points in phase. If one point is taken, and the profile is observed as it passes this point, then the profile is seen to repeat at equal interval of time. The repeat time is the ***period*** T. Otherwise, the period is the time between successive cycles of a repeating sequence of events. ***Frequency*** (υ) is the number of cycles of a repeating sequence of events in a unit interval of time. Frequency, angular frequency and period are reciprocals (or inverses) of one another:

$$T = \frac{2\pi}{\omega} = \frac{1}{\upsilon}. \tag{3.64}$$

The SI units:

$$[\lambda] = \text{m}, \quad [T] = \text{s}, \quad [\upsilon] = \text{Hertz} = \text{Hz} = 1/\text{s} = \text{s}^{-1}, \quad [\omega] = \text{rad/s} = \text{s}^{-1}.$$

Suppose the wave moves from left to right and that a particle at the origin 0 vibrates according to the equation

$$\xi(0,t) = A\cos(\omega t + \alpha), \tag{3.65}$$

where t is the time and $\omega = 2\pi\upsilon = 2\pi/T$.

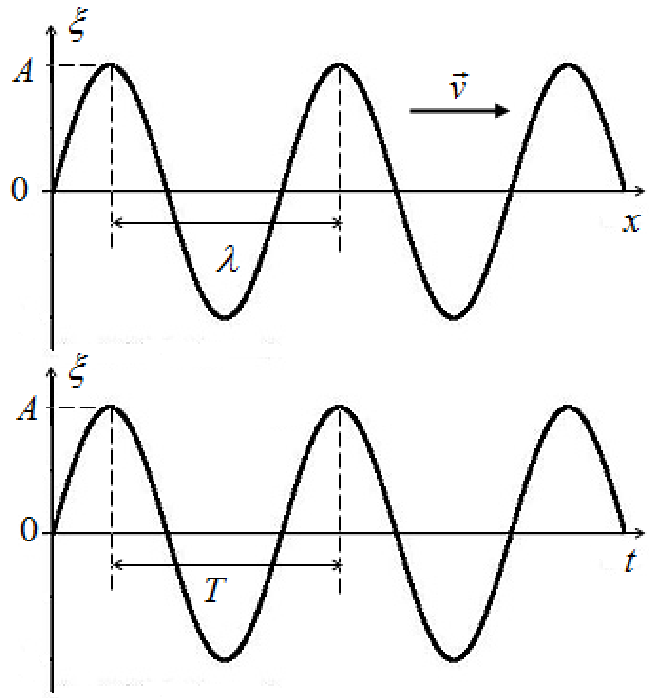


Figure 3.15

For a particle at the distance x from 0 to the right, the phase of the vibration will be different from that at 0, as the time $\tau = x/v$ is necessary for wave to get to the point (x,t). Hence the displacement of any particle at distance x from the origin is given by

$$\xi(x,t) = A\cos\left[\omega(t-\tau)+\alpha\right] = A\cos\left(\omega t - \omega\frac{x}{v} + \alpha\right) = A\cos(\omega t - kx + \alpha).$$

Equation of a ***plane-travelling*** (or plane-progressive) ***wave*** is

$$\xi(x,t) = A\cos\left(\frac{2\pi}{T}t - \frac{2\pi}{\lambda}x + \alpha\right), \tag{3.66}$$

where

$$k = \frac{\omega}{v} = \frac{2\pi\upsilon}{v} = \frac{2\pi}{vT} = \frac{2\pi}{\lambda} \tag{3.67}$$

is the ***wave number***.

Amplitude (*A*) is the maximum magnitude of a periodically varying quantity. Amplitude has the unit of the quantity that is changing (in this case – the displacement).

Phase $\left(\frac{2\pi}{T}t - \frac{2\pi}{\lambda}x + \alpha\right)$ is the stage of development of a periodic process. Two points on a wave with the same phase have the same quantity of disturbance (displacement, etc.) and rate of change of disturbance (velocity, etc.). Phase is an angular quantity. Adjacent points *in phase* are separated by one complete cycle. Adjacent points *out of phase* are separated by half a cycle. The SI unit of phase is the radian.

Generally for a wave propagating in three dimensions, the displacement at point given by position vector $\vec{r}$ at time t is

$$\xi(\vec{r},t) = A\cos\left(\omega t - \vec{k}\vec{r} + \alpha\right), \tag{3.68}$$

where $\vec{k}$ is the ***wave vector***.

The magnitude of the wave vector is equal to the wave number.

The ***speed*** and ***acceleration*** of the vibrating particle may be obtained as the first and the second derivatives of $\xi(x,t)$ with respect to time

$$\dot{\xi} = \frac{\partial\xi}{\partial t} = -A\omega\cdot\sin(\omega t - kx + \alpha), \tag{3.69}$$

$$\ddot{\xi} = \frac{\partial^2\xi}{\partial t^2} = -A\omega^2\cdot\cos(\omega t - kx + \alpha). \tag{3.70}$$

From equations (3.69) and (3.70) the maximum values of the speed and acceleration of the particle are, respectively,

$$\dot{\xi}_{\mathrm{m}} = A\omega , \tag{3.71}$$

$$\ddot{\xi}_{\mathrm{m}} = A\omega^2 . \tag{3.72}$$

The ***difference in phase*** (*phase shift, phase difference*) for two points on the beam of plane-travelling wave separated from each other by the distance Δx is

$$\Delta\varphi = \frac{2\pi}{\lambda}\Delta x , \tag{3.73}$$

where Δx is the ***path-length difference*** or *path-length shift.*

3.2.3. Wave equation

The wave equation is the linear uniform partial differential equation of the second order describing the propagation of the wave in a medium.

Let's receive this equation using the equation of a wave (3.68) assuming the initial phase $\alpha = 0$

$$\xi(\vec{r},t) = A\cos\left(\omega t - \vec{k}\vec{r}\right). \tag{3.74}$$

The first-order and the second-order derivatives of (3.74) with respect to time are

$$\frac{\partial\xi}{\partial t} = -A\omega\cdot\sin(\omega t - kr),$$

$$\frac{\partial^2\xi}{\partial t^2} = -A\omega^2\cdot\cos(\omega t - kr) = -\omega^2\cdot\xi . \tag{3.75}$$

Therefore,

$$\xi = -\frac{1}{\omega^2}\cdot\frac{\partial^2\xi}{\partial t^2}. \tag{3.76}$$

Rewrite the equation (3.74) as

$$\xi(x,y,z,t) = A\cos\left(\omega t - k_x x - k_y y - k_z z\right), \tag{3.77}$$

and find the second partial derivates of it with respect to x:

$$\frac{\partial \xi}{\partial x} = Ak_x \sin\left(\omega t - k_x x - k_y y - k_z z\right),$$

$$\frac{\partial^2 \xi}{\partial x^2} = -Ak_x^2 \cos\left(\omega t - k_x x - k_y y - k_z z\right) = -k_x^2 \cdot \xi \,. \tag{3.78}$$

Similarly to (3.78), the second partial derivatives of (3.77) with respect to y and z are

$$\frac{\partial^2 \xi}{\partial y^2} = -Ak_y^2 \cos\left(\omega t - k_x x - k_y y - k_z z\right) = -k_y^2 \cdot \xi \,, \tag{3.79}$$

$$\frac{\partial^2 \xi}{\partial z^2} = -Ak_z^2 \cos\left(\omega t - k_x x - k_y y - k_z z\right) = -k_z^2 \cdot \xi \,. \tag{3.80}$$

The sum of the second derivatives (3.78), (3.79) and (3.80) may be written taking into account (3.76) as

$$\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 \xi}{\partial z^2} = -\left(k_x^2 + k_y^2 + k_z^2\right)\xi = -k^2 \cdot \xi = k^2 \cdot \frac{1}{\omega^2} \cdot \frac{\partial^2 \xi}{\partial t^2} = \frac{1}{v^2} \cdot \frac{\partial^2 \xi}{\partial t^2} \,. \tag{3.81}$$

Introducing the Laplace operator or Laplacian Δ, we rewrite (3.81) in form of

$$\Delta \xi = \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 \xi}{\partial z^2} \,. \tag{3.82}$$

The ***wave equation*** is

$$\Delta \xi = \frac{1}{v^2} \cdot \frac{\partial^2 \xi}{\partial t^2} \,. \tag{3.83}$$

The wave equation for the one-dimensional wave is

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{1}{v^2} \cdot \frac{\partial^2 \xi}{\partial t^2} \,. \tag{3.84}$$

3.2.4. Energy transferred by a wave. Umov's vector

Waves that propagate in medium transfer energy from one place to another. This energy consists of the kinetic energy of vibrating particles and the potential energy of the deformed areas of the medium. The energy that is transferred by a wave through some area per unit of time is called an ***energy flux*** through this surface

$$\Phi = \frac{dW}{dt}. \tag{3.85}$$

$$[\Phi] = \mathrm{J \cdot s^{-1} = W}.$$

Energy flux density or ***intensity*** (I) of a wave at a place is the energy per second flowing through one square meter held normally at that place in the direction along which the wave travels, i. e., the intensity of any wave is the time averaged rate at which it transmits the energy per unit area through the region of space.

$$I = \frac{dW}{S \cdot dt}. \tag{3.86}$$

$$[I] = \mathrm{J \cdot m^{-2} \cdot s^{-1} = W \cdot m^{-2}}.$$

Suppose that displacement of the wave-front area moving at a speed v for the time interval Δt is l.

If w is ***energy density*** (i. e. average energy of the particles in unit volume), the energy transferred through any area per time Δt is $\Delta W = w \cdot S \cdot v \cdot \Delta t$, and the *intensity* of a wave is

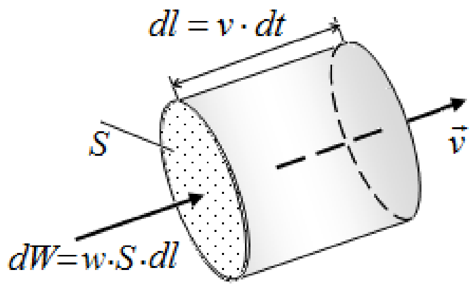


Figure 3.16

$$I = \frac{w \cdot S \cdot v \cdot \Delta t}{S \cdot \Delta t} = w \cdot v. \tag{3.87}$$

The vector form of (3.87) gives the *Umov's vector*

$$\vec{I} = w \cdot \vec{v}, \tag{3.88}$$

which is perpendicular to the wave-front of travelling wave and indicates the direction of wave propagation. Its magnitude is equal to intensity (energy flux density).

If the energy of each particle is $\frac{m\omega^2 A^2}{2}$ and the number of particles in the unit volume is *n*, the energy density w and intensity I are, respectively,

$$w = n \cdot \frac{m\omega^2 A^2}{2} = \frac{\rho\omega^2 A^2}{2}, \tag{3.89}$$

$$I = \frac{\rho\omega^2 A^2}{2} \cdot v, \tag{3.90}$$

where ρ is the density of the medium.

3.2.5. Superposition of waves. Interference

The ***superposition principle***: when two or more waves travel simultaneously through the same medium, each wave proceeds independently as though no other waves were present, and the resultant displacement of any particle is the vector sum of the displacements that the individual waves acting alone would give.

Two waves of the same frequency (or wavelength) having the same phase or a fixed phase difference which remains constant with time are called *coherent waves*. When two or more coherent waves overlap, the phenomenon of *interference* occurs.

Consider two waves travelling from two closely located sources that excite oscillations at the point considerably distant from them. These oscillations are the composition of two unidirectional oscillations. Thus, the amplitude according to (3.29) is

$$A^2 = A_1^2 + A_2^2 + 2A_1A_2 \cos\Delta\varphi. \tag{3.91}$$

Intensity *I* of waves is proportional to A^2, therefore,

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos\Delta\varphi. \tag{3.92}$$

If $A_1 = A_2$, then $I_1 = I_2$, consequently,

$$I = 2I_0 + 2I_0 \cos\Delta\varphi = 2I_0\left(1 + \cos\Delta\varphi\right). \tag{3.93}$$

The intensity at the points, where two waves come, depends on their path-length difference that in its turn depends on the cosine of a phase difference $\cos\Delta\varphi$. As a result, two waves may reinforce each other (*constructive interference*) or they may cancel the effects of each other (*destructive interference)*

1. If $\cos\Delta\varphi = 1$, $I = 2I_0 + 2I_0\cos\Delta\varphi = 4I_0$.

The phase difference is $\Delta\varphi = \pm 2\pi k$, therefore, $\Delta\varphi = \frac{2\pi}{\lambda}\Delta = \pm 2\pi k$ and the *condition for the* ***constructive interference*** (*interference maximum*) is

$$\Delta = \pm k\lambda = \pm 2k\frac{\lambda}{2}. \tag{3.94}$$

2. If $\cos\Delta\varphi = -1$, $I = 2I_0 - 2I_0\cos\Delta\varphi = 0$.

The phase difference is $\Delta\varphi = \pm(2k+1)\pi$, therefore, $\Delta\varphi = \frac{2\pi}{\lambda}\Delta = \pm(2k+1)\pi$ and the *condition for the* ***destructive interference*** (*interference minimum*) is

$$\Delta = \pm(2k+1)\frac{\lambda}{2}. \tag{3.95}$$

Interference is the phenomenon of redistribution of intensity at the superposition of coherent waves.

3.2.6. Standing waves

Sometimes when you vibrate a string or cord it’s possible to get it to vibrate in such a manner that you're generating a wave, but the wave doesn't propagate. It just sits there vibrating up and down in place. Such a wave is called a ***standing wave*** *(or stationary waves)*.

Standing waves can be formed under a variety of conditions, but they are easily demonstrated in a medium which is finite or bounded. The example of finite media are a guitar string (it runs from fret to bridge), a drum head (it's bounded by the rim), the air in a room (it's bounded by the walls), the water in lake (it's bounded by the shores), or the surface of the Earth (although not bounded, the surface of the Earth is finite). In general, standing waves can be produced by any two identical (with equal amplitudes and wavelengths) waves travelling in opposite directions. In a bounded medium, standing waves occur when a wave meets its reflection. The interference of these two waves produces a resultant wave that does not appear to move.

The equations of two plane-travelling waves propagating along an axis x in opposite directions are

$$\begin{cases} \xi_1 = A\cos(\omega t - kx + \alpha_1), \\ \xi_2 = A\cos(\omega t + kx + \alpha_2). \end{cases} \tag{3.96}$$

According to principle of superposition, the resultant displacement is

$$\xi = \xi_1 + \xi_2 = 2A\cos\left(kx + \frac{\alpha_2 - \alpha_1}{2}\right) \cdot \cos\left(\omega t + \frac{\alpha_2 + \alpha_1}{2}\right). \tag{3.97}$$

Let's choose a reference point on x-axis to make $\alpha_2 - \alpha_1 = 0$ and on t-axis to make $\alpha_2 + \alpha_1 = 0$. Then the ***equation of a standing wave*** is

$$\xi = |2A\cos kx| \cdot \cos\omega t . \tag{3.98}$$

At each point of the standing wave, the simple harmonic motion with the same frequency equaled to the frequency of waves occurs (fig. 3.17).

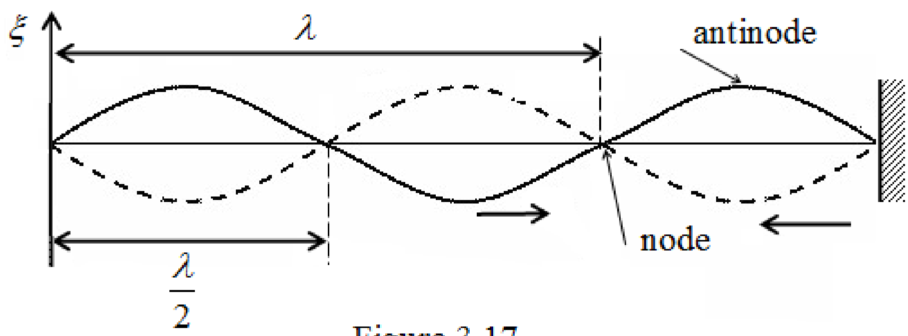


Figure 3.17

The ***amplitude*** of the standing wave $|2A\cos kx|$ depends on coordinate.

1) At points where $kx = \frac{2\pi}{\lambda}x = \pm\pi m$, $m = 0,1,2,...$, the value of $\cos kx = \pm 1$, and amplitude is $|2A\cos kx| = |2A|$.

The points where the resultant amplitude is maximum are called **antinodes** (or loops). Coordinates of antinodes are

$$x_{\text{antinodes}} = \pm m\frac{\lambda}{2}, \tag{3.99}$$

2) At points where $kx = \frac{2\pi}{\lambda}x = \pm(2m+1)\frac{\pi}{2}$, $m = 0,1,2,...$, the value of $\cos kx = 0$, the amplitude is $|2A\cos kx| = 0$.

The points where the resultant amplitude is zero are called ***nodes***. Coordinates of nodes are

$$x_{\text{nodes}} = \pm\left(m + \frac{1}{2}\right)\frac{\lambda}{2}. \tag{3.100}$$

The distance between any two successive antinodes or nodes is equal to $\frac{\lambda}{2}$ and the distance between an antinode and a node is $\frac{\lambda}{4}$.

.There is no energy transfer in the standing wave. The total energy of oscillations of each element of volume of the medium limited by the adjacent node and antinode does not depend on time. It only periodically transforms from a kinetic energy concentrated basically close to antinode into a potential energy of elastically deformed medium (near the node). Lack of energy transfer is the result of the fact that two identical waves travelling in opposite directions transfer an equal energy.

3.2.7. Standing waves in strings and pipes

1. In musical instruments like sitar, violin, etc. sound is produced due to the vibrations of the stretched strings. When a string under tension is set into vibration, a transverse progressive wave moves towards the end of the wire and gets reflected. Thus standing waves are formed.

When a wire AB of length l is made to vibrate in one segment (fig. 3.18, *a*) then $l = \frac{\lambda_1}{2}$ and $\lambda_1 = 2l$. This gives the lowest frequency called ***fundamental frequency***

$$\upsilon_1 = \frac{v}{\lambda_1} \; . \qquad (3.101)$$

If the wire AB is made to vibrate in two segments (fig. 3.18, *b*) then $l = \lambda_2$ and frequency

$$\upsilon_2 = \frac{v}{\lambda_2} \; . \qquad (3.102)$$

is the frequency of the first overtone. Since the frequency is equal to twice the fundamental, it is also known as ***second harmonic***. Similarly, higher overtones are produced, if the wire vibrates with more segments. If there are n segments, the length of each segment is $l = \frac{\lambda_n}{2} n$ and the wavelength is $\lambda_n = \frac{2l}{n}$.

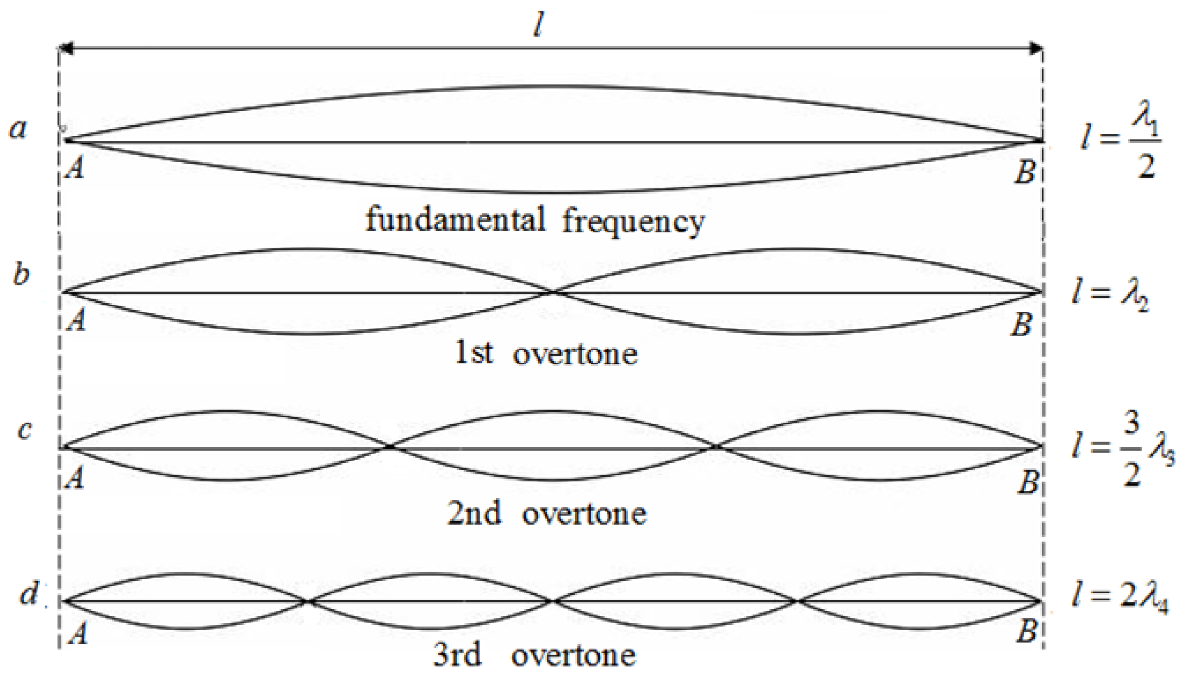


Figure 3.18

Frequency

$$\upsilon_n = n\upsilon_1 \qquad (3.103)$$

is the n-th harmonic corresponding to $(n-1)$-th overtone.

2. Musical wind instruments like flute, clarinet etc. are based on the principle of vibrations of air columns. Due to the superposition of the incident wave and the reflected wave, longitudinal stationary waves are formed in the pipe.

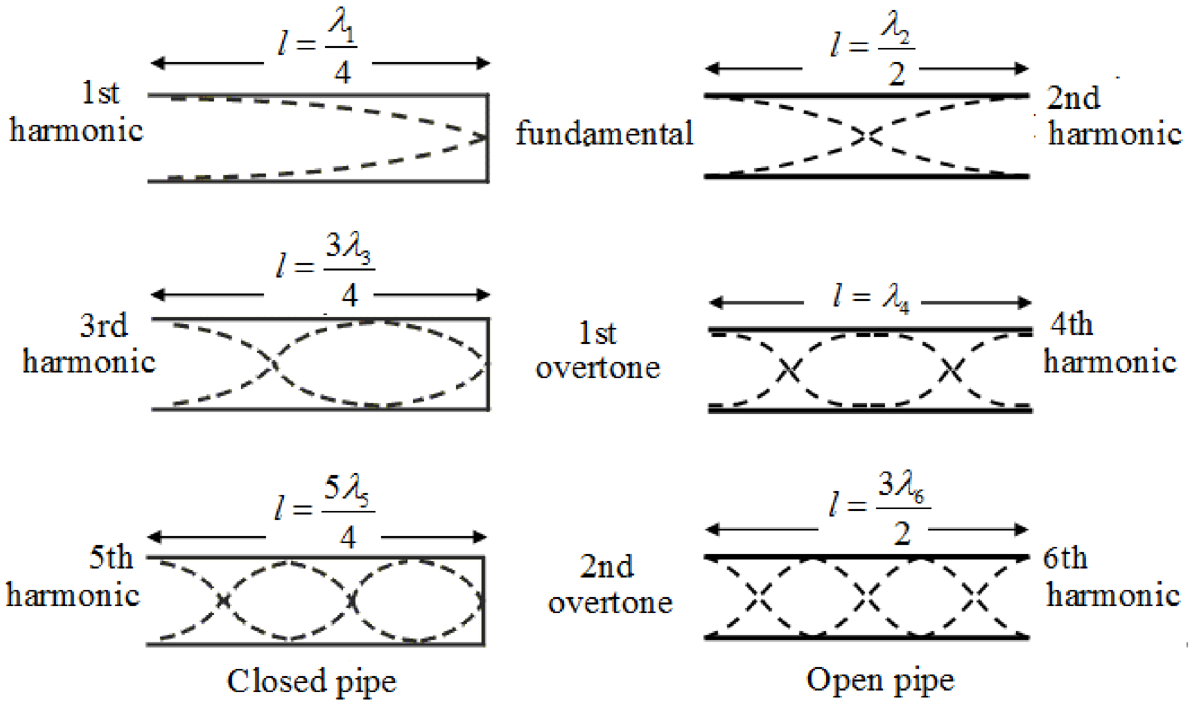


Figure 3.19

Organ pipes are musical instruments which are used to produce musical sound by blowing air into the pipe. Organ pipes are two types (a) closed pipes (closed at one end), (b) open pipes (open at both ends) (fig. 3.19).

a) ***Closed pipe***. There is a node at the closed end and an antinode at the open end.

The *fundamental* frequency of the vibrations in the closed pipe is

$$\upsilon_1 = \frac{v}{\lambda_1} = \frac{v}{4l}. \tag{3.104}$$

The frequency of the first overtone (or the third harmonic) is

$$\upsilon_3 = \frac{v}{\lambda_3} = \frac{3v}{4l} = 3\upsilon_1 .$$

The frequency of n-th overtone or $(2n+1)$-th harmonic is

$$\upsilon_n = (2n+1)\upsilon_1 . \tag{3.105}$$

In a closed pipe only odd harmonics are produced. The frequencies of harmonics are in the ratio of 1 : 3 : 5….

b) ***Open pipe***. Antinodes are formed at the ends and a node is formed in the middle of the pipe. The *fundamental* frequency is

$$\upsilon_1 = \frac{v}{2l}. \qquad (3.106)$$

For the first overtone or the second harmonic the frequency is

$$\upsilon_2 = \frac{v}{l} = 2\upsilon_1.$$

The frequency of the n-th overtone for open pipe is

$$\upsilon_n = (n+1)\upsilon_1. \qquad (3.107)$$

The frequencies of harmonics are in the ratio 1: 2: 3:..

3.2.8. The Doppler effect

If the observer or the source or both are in motion then the observer notes an apparent change in frequency from actual frequency of the wave emitted by the source. This phenomenon is called the ***Doppler effect*** and the difference between the actual and detected frequencies is known as ***Doppler shift***. This effect was first noted by Austrian physicist Christian Andreas Doppler (1803–1853) in 1842.

The Doppler effect holds both for sound waves and for electromagnetic waves, including microwaves, radio waves, and visible light. Here, however, we consider only sound waves, and we take as a reference frame the body of air through which these waves travel. As a result, the speeds of a source *S* of sound waves (v_S) and of a detector *D* of those waves (v_D) are measured *relatively to the body of air*. In other words, the body of air is stationary relative to the ground, so the speeds can also be measured relative to the ground. Also we assume that *S* and *D* move either directly toward or directly away from each other, at speeds less than the speed of sound (fig. 3.20).

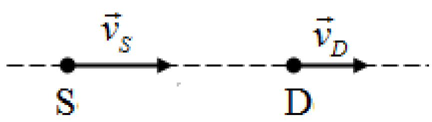


Figure 3.20

When both the source (S) and the detector (D) are in motion along the same straight line, the moving detector will receive a wave whose apparent (detected) frequency υ' depends on the actual frequency of the source υ as

$$\upsilon' = \upsilon \frac{v \pm v_D}{v \mp v_S}, \tag{3.108}$$

where v is the speed of sound through the air (upper signs if "toward", lower signs if "away").

Approaching each other corresponds to

$$\upsilon' = \upsilon \frac{v + v_D}{v - v_S}, \tag{3.109}$$

and the moving away from each other relates to

$$\upsilon' = \upsilon \frac{v - v_D}{v + v_S}. \tag{3.110}$$

Let us consider the following *special cases*.

1. If the source is at rest ($v_S = 0$) and the detector is moving away from the source, the observed frequency is less than the actual frequency, $\upsilon' < \upsilon$.

$$\upsilon' = \upsilon \frac{v - v_D}{v}. \tag{3.111}$$

2. If the source is at rest ($v_S = 0$) and the detector is moving towards the source, the observed frequency is more than the actual frequency, $\upsilon' > \upsilon$.

$$\upsilon' = \upsilon \frac{v + v_D}{v}. \tag{3.112}$$

3. If the detector is at rest ($v_D = 0$) and the source is moving towards the detector, the observed frequency is more than the actual frequency, $\upsilon' > \upsilon$.

$$\upsilon' = \upsilon \frac{v}{v - v_S}. \tag{3.113}$$

3. If the detector is at rest ($v_D = 0$) and the source is moving away from the detector, the observed frequency is less than the actual frequency, $\upsilon' > \upsilon$.

$$\upsilon' = \upsilon \frac{v}{v + v_S}. \tag{3.114}$$

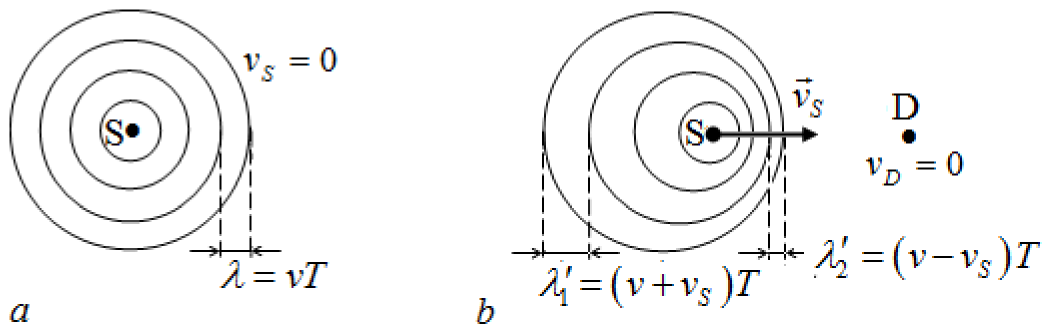


Figure 3.21

The Doppler effect is observed because the distance between the source of sound and the observer is changing. If the source and the observer are approaching, then the distance is decreasing and if the source and the observer are receding, then the distance is increasing. The source of sound always emits the same frequency. Therefore, for the same period of time, the same number of waves must fit between the source and the observer. If the distance is large, then the waves can be spread apart; but if the distance is small, the waves must be compressed into the smaller distance (fig. 3.21). For these reasons, if the source is moving towards the observer, the observer perceives sound waves reaching him or her at a more frequent rate (high pitch). And if the source is moving away from the observer, the observer perceives sound waves reaching him or her at a less frequent rate (low pitch).

The Doppler effect can be observed for any type of wave – water wave, sound wave, light wave, etc. We are most familiar with the Doppler effect because of our experiences with sound waves. For example, a police car or emergency vehicle that is traveling towards the observer on the highway. As the car approaching with its siren blasting, the pitch of the siren sound (a measure of the siren's frequency) is high; and then suddenly after the car passed by, the pitch of the siren sound is low. That is the Doppler effect – an apparent shift in frequency for a sound wave produced by a moving source.

The examples of the practical applications of the Doppler effect are following. The microwave Doppler radar is used in aircraft to measure its flight speed. The Doppler sonar (**So**und **N**avigation **A**nd **R**anging) can measure the cruising speed of an oceanographic vehicle relative to the seafloor or to given water mass. In the areas of oceanographic research and operational oceanography, this effect is extensively used for remote measurements of surface and subsurface oceanic circulation. The Doppler effect for electromagnetic waves is of intense interest to astronomers who use the information about the shift in frequency of electromagnetic waves produced by moving stars in our galaxy and beyond in order to derive information about those stars and galaxies. From an operational point of view, the Doppler effect is applied in modern traffic control to detect speeding motorists. This effect is also used in sports, through Doppler radar systems to measure ball speed. An echocardiogram can produce accurate assessment of the direction of blood flow and the velocity of blood and cardiac tissue at any arbitrary point using the Doppler effect. Velocity measurement of blood flow in arteries and veins based on this effect is an effective tool for diagnosis of vascular problems in medicine.

PROBLEMS

Problem 3.1

If a particle undergoes SHM $x = 0.2\sin 2\pi t$ *(m), what is the total distance it travels in one period? Find the angular frequency and period of oscillation.*

Solution

The particle would travel four times the amplitude: from $x = 0$ to $x = A$; then to $x = 0$, then to $x = -A$, and to $x = 0$. So the total distance is equal to $4A = 4 \cdot 0.2 = 0.8$ m.

The angular velocity is $\omega_0 = 2\pi$, and the period of oscillations is

$$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{2\pi} = 1 \text{ s.}$$

Problem 3.2

A body oscillates with the simple harmonic motion according to the equation $x = 6\cos\left(3\pi t + \frac{\pi}{3}\right)$ *(m). Calculate the displacement, the velocity, the acceleration, and the phase at the time* t *= 2 s. Find also the angular velocity, frequency, and the period of motion.*

Solution

The phase is

$$3\pi t + \frac{\pi}{3}\bigg|_{t=2s} = 3\pi \cdot 2 + \frac{\pi}{3} = \frac{19\pi}{3} = 19.9 \text{ rad.}$$

The displacement is

$$x = 6\cos\left(3\pi t + \frac{\pi}{3}\right)\bigg|_{t=2s} = 6\cos\left(3\pi \cdot 2 + \frac{\pi}{3}\right) = 6\cos\left(\frac{19\pi}{3}\right) = 3 \text{ m.}$$

The velocity is

$$v = \frac{dx}{dt} = -6 \cdot 3\pi \cdot \sin\left(3\pi t + \frac{\pi}{3}\right)\bigg|_{t=2s} = -18\pi \sin\left(\frac{19\pi}{3}\right) = -49 \text{ m/s.}$$

The acceleration is

$$a=\frac{dv}{dt}=-6\cdot(3\pi)^2\cdot\cos\left(3\pi t+\frac{\pi}{3}\right)\Bigg|_{t=2s}=-54\pi^2\cos\left(\frac{19\pi}{3}\right)=-266.5\ \text{m/s}^2.$$

The angular velocity is $\omega_0=3\pi$ (rad/s).

Since $\omega_0=2\pi\upsilon_0$, the frequency is $\upsilon_0=\frac{\omega_0}{2\pi}=\frac{3\pi}{2\pi}=1.5\ \text{Hz}$.

The period $T=\frac{2\pi}{\omega_0}=\frac{1}{\upsilon_0}=\frac{2}{3}$ s.

Problem 3.3

For what part of the period the oscillating point displaces by the half of amplitude, if it started from the equilibrium position?

Solution

Let us choose the trigonometric function for the oscillations description taking into account the given data. The point starts from the equilibrium position, then its displacement on the time instant $t=0$ is equal to $x=0$, therefore, the equation of oscillating motions is

$$x=A\sin\frac{2\pi}{T}t.$$

Substituting of the displacement of a point from the mean position $x=\frac{A}{2}$ in the equation of oscillations gives

$$\frac{A}{2}=A\sin\frac{2\pi}{T}t,$$

$$\sin\frac{2\pi}{T}t=\frac{1}{2}.$$

$$\frac{2\pi}{T}t=\arcsin\frac{1}{2}=\frac{\pi}{6}.$$

Hence, the required time is $t=\frac{T}{12}$.

Problem 3.4

A point oscillates according to the dependence $x = 5\cos\omega_0 t$ *(m), where* $\omega_0 = 2\ s^{-1}$*. Find the acceleration of the point when its speed is equal to* 8 *m/s.*

Solution

The speed and accelerations of the vibrating point depends on time as

$$v = \dot{x} = -A\omega_0 \sin\omega_0 t,$$

$$a = \ddot{x} = -A\omega_0^2 \cos\omega_0 t.$$

From the first equation

$$\sin\omega_0 t = -\frac{v}{A\omega_0}.$$

For calculation of the acceleration we need to know the magnitude of $\cos\omega_0 t$, rather than the instant of time t. Therefore, using the trigonometric identity $\sin^2\omega t + \cos^2\omega t = 1$ the value of $\cos\omega_0 t$ may be found as

$$\cos\omega_0 t = \sqrt{1 - \sin^2\omega_0 t} = \sqrt{1 - \left(v/A\omega_0\right)^2}.$$

Consequently,

$$a = -A\omega_0^2 \cos\omega_0 t = -A\omega_0^2 \sqrt{1 - \left(v/A\omega_0\right)^2}, \text{ and}$$

$$a = -5 \cdot 2^2 \sqrt{1 - \left(8/5 \cdot 2\right)^2} = -12\,\text{m/s}^2.$$

Problem 3.5

The maximum speed and acceleration of a particle executing simple harmonic motion are 10 *cm/s and* 50 *cm/s. Find the positions of the particle when the speed is* 8 *cm/s, if* $x(0) = 0$.

Solution

The maximum speed and accelerations are equal to

$$\begin{cases} v_{\max} = A\omega_0, \\ a_{\max} = A\omega_0^2. \end{cases}$$

Divide the second equation by the first one and obtain

$$\frac{a_{\max}}{v_{\max}} = \frac{A\omega_0^2}{A\omega_0} = \omega_0 = \frac{50}{10} = 5 \text{ rad/s.}$$

The amplitude of the oscillations calculated using the first equation is

$$A = \frac{v_{\max}}{\omega_0} = \frac{0.1}{5} = 0.02 \text{ m.}$$

The given data $x(0) = 0$ shows that the equation of particle motion has to be written as $x = A\sin\omega_0 t$, and the speed depends on time as

$$v = A\omega_0 \cos\omega_0 t .$$

Therefore,

$$\cos\omega_0 t = \frac{v}{A\omega_0} = \frac{0.08}{0.02 \cdot 5} = 0.8 ,$$

and

$$\sin\omega_0 t = \pm\sqrt{1-\cos^2\omega_0 t} = \pm\sqrt{1-0.8^2} = \pm 0.6 .$$

Then the desired position is

$$x = A\sin\omega_0 t = \pm 0.02 \cdot 0.6 = \pm 0.012 \text{ m.}$$

Problem 3.6

The maximum velocity of oscillating point is 10 *cm/s, and its maximum acceleration is* 100 *m/s*^{2}*. Find the angular frequency and the amplitude of oscillations.*

Solution

The equations that describe the simple harmonic motion are

$$x = A\cos(\omega_0 t + \alpha) ,$$

$$v = -A\omega_0 \sin(\omega t + \alpha) = -v_{\max}\sin(\omega_0 t + \alpha) ,$$

$$a = -A\omega_0^2 \cos(\omega_0 t + \alpha) = -a_{\max}\cos(\omega_0 t + \alpha) .$$

Then the maximum magnitudes of displacement, velocity and acceleration are

$$\begin{cases} x_{\max} = A, \\ v_{\max} = A\omega_0, \\ a_{\max} = A\omega_0^2. \end{cases}$$

Dividing the third equation of system by the second equation we obtain

$$\frac{a_{\max}}{v_{\max}} = \frac{A\omega_0{}^2}{A\omega_0} = \omega_0, \qquad \text{and} \qquad \omega_0 = 10\,\text{s}^{-1}.$$

The period and the amplitude of oscillations are, respectively,

$$T = \frac{2\pi}{\omega_0} = 0,2\pi = 0,628 \text{ s}, \qquad A = \frac{v_{\max}}{\omega_0} = \frac{0,1}{10} = 0,01 \text{ m}.$$

Problem 3.7

The equation of motion of a particle started at $t = 0$ is given by $x = 5\sin\left(20t + \frac{\pi}{3}\right)$ (cm). When does the particle a) first come to rest; b) first have zero acceleration; and c) first have maximum speed?

Solution

a) If the speed of the particle is described by equation

$$v = \frac{dx}{dt} = 5 \cdot 20 \cdot \cos\left(20t + \frac{\pi}{3}\right),$$

at $v = 0$, the equation is

$$\cos\left(20t + \frac{\pi}{3}\right) = 0.$$

It gives

$$20t + \frac{\pi}{3} = \frac{\pi}{2}.$$

$$20t = \frac{\pi}{6}.$$

$$t = \frac{\pi}{120} = 0.026 \text{ s.}$$

b) The acceleration of the particle is

$$a = \frac{dv}{dt} = -5 \cdot 20^2 \cdot \sin\left(20t + \frac{\pi}{3}\right).$$

$$\sin\left(20t + \frac{\pi}{3}\right) = 0,$$

Firstly after the beginning of the oscillation process acceleration becomes zero when phase is equal to π:

$$20t + \frac{\pi}{3} = \pi,$$

$$t = \frac{\pi}{30} = 0.105 \text{ s.}$$

c) $v = 5 \cdot 20 \cdot \cos\left(20t + \frac{\pi}{3}\right).$

The speed is maximum when $\cos\left(20t + \frac{\pi}{3}\right) = 1$, therefore,

$$20t + \frac{\pi}{3} = \pi.$$

$$t = \frac{\pi}{30} = 0.105 \text{ s.}$$

Note, that we obtained the same result as in (b). It means that when the particle passes the equilibrium position it moves at its maximum speed and has zero acceleration.

Problem 3.8

At $t=0$, the displacement of the point in a linear oscillator is $x(0)=-8.6$ cm, its velocity $v(0)=-0.93$ m/s and its acceleration $a(0)=48$ m/s². What are the angular frequency ω_0 and the frequency υ_0? What is the phase constant? What is the amplitude of the motion?

Solution

The displacement of the particle is given by

$$x(t)=A\cos(\omega_0 t+\alpha).$$

Hence,

$$x(0)=A\cos\alpha=-8.6\ \text{cm}=-\,0.086\ \text{m},$$

$$v(0)=-\omega_0 A\sin\alpha=0.93\ \text{m/s},$$

$$a(0)=-\omega_0^2 A\cos\alpha=48\ \text{m/s}^2.$$

Thus,

$$\frac{a(0)}{x(0)}=\frac{-\omega_0^2 A\cos\alpha}{A\cos\alpha}=-\omega_0^2,$$

$$\omega_0=\sqrt{-\frac{a(0)}{x(0)}}=\sqrt{-\frac{48}{-0.086}}=23.62\ \text{rad/s},$$

$$\upsilon_0=\frac{\omega_0}{2\pi}=\frac{23.62}{2\pi}=3.76\ \text{Hz}.$$

$$\frac{v(0)}{x(0)}=\frac{-\omega_0\cancel{A}\sin\alpha}{\cancel{A}\cos\alpha}=-\omega_0\cdot\tan\alpha\,,$$

$$\tan\alpha=-\frac{v(0)}{\omega_0 x(0)}=-\frac{0.93}{23.62\cdot 0.086}=-0.458\,.$$

Hence $\alpha=155.4°$ and $335.4°$ in the range $0\le\alpha\le 2\pi$. We shall see below how to choose between the two values.

$$A=\frac{x(0)}{\cos\alpha}=\frac{-0.086}{\cos\alpha}.$$

The amplitude of the motion is a positive constant. So, α = 335.4° cannot be correct phase, as $\cos 334.5° = 0.909$ and the amplitude turns out negative. Therefore, $\alpha = 155.4^0$, as a result $\cos 155.4^0 = -0.909$ gives positive amplitude equaled to

$$A = \frac{-0.086}{-0.909} = 0.0946 \text{ m.}$$

Problem 3.9

A point moves with simple harmonic motion whose period is 4 *s. If it starts from rest at a distance* 4 *cm from the centre of its path, find the time that elapses before it has described* 1 *cm and the velocity it has then acquired. How long will the point take to reach the center of its path?*

Solution

Amplitude of the oscillations is $A = 4$ cm, the time period is $T = 4$ s, and the angular speed is

$$\omega_0 = 2\pi/T = 2\pi/4 = \pi/2 \text{ rad/s.}$$

If the particle is at a point $x = 4$ cm from the mean position when $t = 0$, the equation of its motion is $x = A\cos\omega_0 t$.

After covering 1 cm-distance the displacement from equilibrium position is $x = 4 - 1 = 3$ cm. As a result,

$$\cos\omega_0 t = \frac{x}{A} = \frac{3}{4} = 0.75.$$

$$\omega_0 t = \arccos 0.75 = 0.72 \text{ rad.}$$

The time is

$$t = \frac{0.72}{\omega_0} = \frac{0.72 \cdot T}{2\pi} = \frac{0.72 \cdot 4}{2 \cdot 3.14} = 0.46 \text{ s.}$$

The speed of the particle is

$$v = -A\omega_0 \sin\omega_0 t = -A\omega_0\sqrt{1 - \cos^2\omega_0 t} = -A\omega_0\sqrt{1 - \frac{x^2}{A^2}} = -\omega_0\sqrt{A^2 - x^2} \ .$$

Substitution gives

$$v = -\omega_0\sqrt{A^2 - x^2} = -\frac{\pi}{2}\sqrt{4^2 - 3^2} = -4.2 \text{ cm/s}.$$

The negative sign denotes that the particle located at the positive position ($x > 0$) moves to the mean position "from right to the left".

At the midpoint $x = 0$, therefore,

$$0 = A\cos\omega_0\tau \ ,$$

$$\cos\omega_0\tau = 0, \ \omega_0\tau = \pi/2,$$

$$\tau = \frac{\pi}{\omega_0} = \frac{\pi\cdot 2}{2\cdot\pi} = 1 \text{ s}.$$

Problem 3.10

A particle executing SHM on a straight line has a velocity of 4 *cm/s when at a distance of* 3 *m from the mean position, and 3 m/s, when at a distance of 4 m from it. Find the time it takes to travel* 2 *m from the positive extremity of its oscillation.*

Solution

The velocity of the particle executing SHM is derivative of displacement x with respect to time t, consequently, if displacement is $x = A\sin(\omega_0 t + a)$, then

$$v = \frac{dx}{dt} = A\omega_0\cos(\omega_0 t + \alpha) = A\omega_0\sqrt{1 - \sin^2(\omega_0 t + \alpha)} = \omega_0\sqrt{A^2 - A^2\sin^2(\omega_0 t + \alpha)}$$

$$v = \omega_0\sqrt{A^2 - x^2} \ .$$

Substituting the given data in this expression gives

$$\begin{cases} 4^2 = \omega_0^2\left(A^2 - 3^2\right) \\ 3^2 = \omega_0^2\left(A^2 - 4^2\right) \end{cases}$$

On solving them, we get:

$$A = 5 \text{ m}, \ \omega_0 = 1 \text{rad/s}.$$

For the movement from the positive extremity through a distance 2 m, the displacement (from the mean point) is

$$x = 5 - 2 = 3 \text{ m}.$$

We use the equation

$$x = A\cos\omega_0 t,$$

because the oscillation process begins from the extremity point, i. e., at $t = 0$, $x = A$.

Using this expression, we obtain

$$\cos\omega_0 t = \frac{x}{A} = \frac{3}{5} = 0.6,$$

$$\omega_0 t = \arccos 0.6 = 53.1^\circ = 0.3\pi \text{ rad},$$

$$t = \frac{0.3\pi}{\omega_0} = \frac{0.3\cdot 3.14}{1} = 0.94 \text{ s}.$$

Problem 3.11

A particle is simultaneously subjected to two simple harmonic motions in the same direction, each of frequency 5 *Hz. If the amplitudes are* 0.05 *m and* 0.02 *m respectively, and phase difference between them is* 45°*, find the amplitude of the resultant displacement and its phase relative to the first component. Write down the expression for the resultant displacement as a function of time.*

Solution

Let the phase constant (an initial phase) α_1 of the first component be zero, then the phase constant α_2 of the second phase is $\frac{\pi}{4}$; the amplitude of the first motion is $A_1 = 0.05$ m, and the amplitude of the second motion is $A_2 = 0.02$ m.

The amplitude of the resultant motion is given by equation

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\alpha_2 - \alpha_1)}.$$

Thus

$$A = \sqrt{0.05^2 + 0.02^2 + 2\cdot 0.05\cdot 0.02\cos(\pi/4)} = 6.57\cdot 10^{-2} \text{ m}.$$

The phase constant α of the resultant motion is given by equation

$$\tan\alpha = \frac{A_1\sin\alpha_1 + A_2\sin\alpha_2}{A_1\cos\alpha_1 + A_2\cos\alpha_2},$$

$$\tan\alpha = \frac{0.05\sin 0^\circ + 0.02\sin(\pi/4)}{0.05\cos 0^\circ + 0.02\cos(\pi/4)} = 0.22,$$

$$\alpha = 12.4^\circ \approx 0.07\pi \text{ rad.}$$

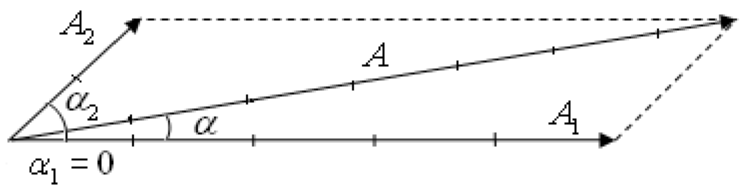
The frequency of each motion is $\upsilon_0 = 5\,\text{Hz}$, therefore, an angular frequency is

$$\omega_0 = 2\pi\upsilon_0 = 10\pi \text{ rad/s.}$$

With these values of A, α and ω, the expression for the resultant displacement takes the form

$$x = 6.57\cdot 10^{-2}\cos(10\pi t + 0.07\pi) \text{ m.}$$

The resultant amplitude A and the initial phase α may be obtained by the method of vector addition of amplitudes. The vector diagram is shown below.



The phase constant of the first component is zero, and of the second component is $\dfrac{\pi}{4}$. Vector A is the resultant of vectors $A_1 = 0.02$ m and $A_2 = 0.05$ m, respectively. The angle α is equal to the phase constant of the resultant motion.

Problem 3.12

The point is executing three SHM of the same direction simultaneously: $x_1 = 3\cos(5\pi t)$, $x_2 = 3\sin\left(5\pi t + \dfrac{\pi}{6}\right)$, $x_3 = 3\sin\left(5\pi t - \dfrac{\pi}{6}\right)$, *where displacements are in centimetres. Find the equation of its resultant motion.*

Solution

This problem may be solved by graphical and analytical methods. But any way the first thing to do is to obtain all equations in identical trigonometric form, for example, using sine functions in all equations. Therefore, transformation of the first equation gives

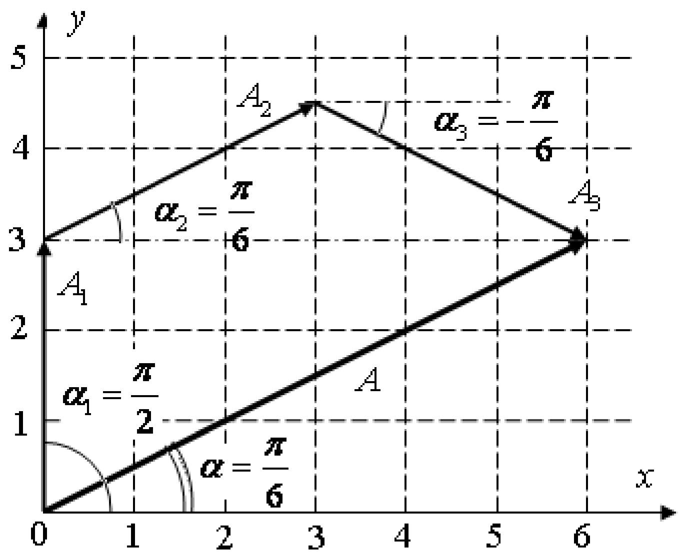
$$x_1 = 3\cos(5\pi t) = 3\sin(5\pi t + \pi/2) \text{ cm.}$$

The first method

Let us construct the diagram using the rule and the graduating arc with scrupulous attention to the scale. An obtained diagram is shown in the figure below.

The length of resultant vector A, measured by the rule, is 6 cm. The initial phase, i. e., the angle α, measured by the graduating arc is $30°$ or $\pi/6$ rad. Therefore, the equation of the resultant oscillations is

$$x = 6\sin\left(5\pi t + \frac{\pi}{6}\right) \text{ cm.}$$



The second method

Lets find the sum of the first two oscillations $x_1 = 3\sin\left(5\pi t + \frac{\pi}{2}\right)$ and $x_2 = 3\sin\left(5\pi t + \frac{\pi}{6}\right)$. The amplitude of the resultant oscillations is

$$A_{12}=\sqrt{A_1^2+A_2^2+2A_1A_2\cos(\alpha_2-\alpha_1)}=\sqrt{9+9+2\cdot 9\cdot\cos\frac{\pi}{3}}=3\sqrt{3}\ \text{cm},$$

and the initial phase is

$$\tan\alpha_{12}=\frac{A_1\sin\alpha_1+A_2\sin\alpha_2}{A_1\cos\alpha_1+A_2\cos\alpha_2}=\frac{3\sin\frac{\pi}{2}+3\sin\frac{\pi}{6}}{3\cos\frac{\pi}{2}+3\cos\frac{\pi}{6}}=\frac{3\cdot 1+3\cdot 0.5}{3\cdot 0+3\cdot 0.866}=1.73,$$

$$\alpha_{12}=\arctan(1.73)=\frac{\pi}{3}.$$

As a result,

$$x_{12}=A_{12}\sin(5\pi t+\alpha_{12})=3\sqrt{3}\sin\left(5\pi t+\frac{\pi}{3}\right)\ \text{cm}.$$

Now let us find the sum of $x_{12}=3\sqrt{3}\sin\left(5\pi t+\frac{\pi}{3}\right)$ and $x_3=3\sin\left(5\pi t-\frac{\pi}{6}\right)$.

$$A=\sqrt{A_{12}^2+A_3^2+2A_{12}A_3\cos(\alpha_{12}-\alpha_3)}=\sqrt{27+9+2\cdot 3\sqrt{3}\cdot\cos\left[\frac{\pi}{3}-\left(-\frac{\pi}{6}\right)\right]}=6$$

$$\tan\alpha=\frac{A_{12}\sin\alpha_{12}+A_3\sin\alpha_3}{A_{12}\cos\alpha_{12}+A_3\cos\alpha_3}=$$

$$=\frac{3\sqrt{3}\sin\frac{\pi}{3}+3\sin\left(-\frac{\pi}{6}\right)}{3\sqrt{3}\cos\frac{\pi}{3}+3\cos\left(-\frac{\pi}{6}\right)}=\frac{3\sqrt{3}\cdot 0.866+3\cdot 0.5}{3\sqrt{3}\cdot 0.5+3\cdot 0.866}=0.577,$$

$$\alpha=\arctan(0.577)=\frac{\pi}{6}.$$

The equation of the resultant oscillation is $x=6\sin\left(5\pi t+\frac{\pi}{6}\right)$ cm.

Problem 3.13

A point mass is subjected to two simultaneous sinusoidal displacement in x*-direction* $x_1 = A\sin\omega_0 t$ *and* $x_2 = A\sin\left(\omega_0 t + \frac{2\pi}{3}\right)$. *Adding a third sinusoidal displacement* $x_3 = B\sin\left(\omega_0 t + \alpha_3\right)$ *brings the mass to a complete rest. Find the values of* B *and* α_3.

Solution

The resultant motion of the particle subjected by two simultaneous simple harmonic motions (SHM) is

$$x_{12} = x_1 + x_2 = A\sin\omega t + A\sin\left(\omega t + \frac{2\pi}{3}\right) = A_{12}\sin\left(\omega t + \alpha_{12}\right).$$

$$A_{12} = \sqrt{A^2 + A^2 - 2A^2\cos\left(\frac{2\pi}{3} - 0\right)} = A,$$

$$\tan\alpha_{12} = \frac{A\sin 0° + A\sin\left(2\pi/3\right)}{A\cos 0° + A\cos\left(2\pi/3\right)} = 1.732,$$

$$\alpha_{12} = \arctan 1.732 = \frac{\pi}{3}.$$

The amplitude of the resultant simple harmonic motion is A_{12} and the initial phase is $\alpha_{12} = \pi/3$.

As the particle remains at rest on adding the third simple harmonic motion, $x_3 = B\sin\left(\omega t + \alpha_3\right)$, the amplitude B of the third SHM must be A itself, but it must be $180°$ (or, π radian) out of phase. In other words, the initial phase α of the third SHM must be $\frac{\pi}{3} + \pi = \frac{4\pi}{3}$.

Problem 3.14

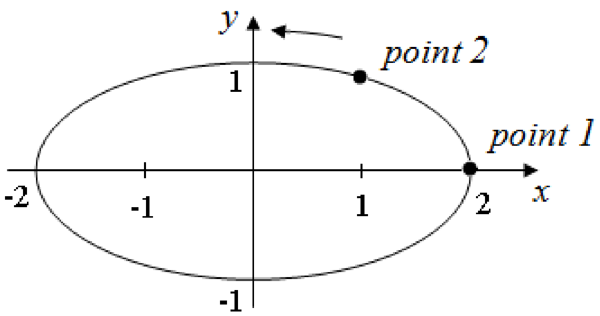
Two vibrations, at right angle to each other, are described by the equations $x = 2\cos\frac{\pi}{3}t$ *(cm) and* $y = \sin\frac{\pi}{3}t$ *(cm). Construct the curve for the combined motion and determine the direction of motion.*

Solution

Firstly we have to obtain both equations in identical trigonometric form. For this purpose we transform the second equation from sine function to cosine:

$$\begin{cases} x = 2\cos\dfrac{\pi}{3}t, \\ y = \cos\left(\dfrac{\pi}{3}t - \dfrac{\pi}{2}\right). \end{cases}$$

The phase difference between two oscillations is equal to $\Delta\alpha = \dfrac{\pi}{2}$, therefore, the equation of trajectory is



$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1,$$

$$\frac{x^2}{2^2} + \frac{y^2}{1^2} = 1.$$

Thus, the trajectory of the particle is an ellipse with semi major and semi minor axes A and B, coinciding with x-and y-axes, respectively, i. e., it is an ellipse reduced to the principal axes (see figure).

The direction of rotation (clockwise or anticlockwise) of the particle may be obtained from the x- and y-motions of the particle when t is increased gradually.

Let's find the coordinates of the particle for two close instants of time t_1 and t_2. To estimate their closeness it is necessary to compare $\Delta t = t_1 - t_2$ with the period of oscillations T

$$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{\pi/3} = 6 \text{ s}.$$

Than we may take $\Delta t = 1$ s, and for

$$t_1 = 0: \quad \begin{cases} x_1 = 2\cos 0 = 2 \\ y_1 = \cos\left(-\dfrac{\pi}{2}\right) = 0 \end{cases} \quad \Rightarrow \text{ point 1 } (2;0),$$

$$t_2 = 1\,\text{s}: \quad \begin{cases} x_2 = 2\cos\dfrac{\pi}{3} = 1 \\ y_2 = \cos\left(\dfrac{\pi}{3} - \dfrac{\pi}{2}\right) = 0.86 \end{cases} \quad \Rightarrow \text{ point 2 } \ (1;0.86).$$

Hence, the particle moves in counterclockwise direction.

Problem 3.15

Two oscillations, at right angle to each other, are described by the equations $x = 0.02\sin\pi t$ *(m) and* $y = 0.01\cos\left(\pi t + \dfrac{\pi}{2}\right)$ *(m). Construct the trajectory for the combined motion.*

Solution

First it is necessary to transform the second equation to the sin-form

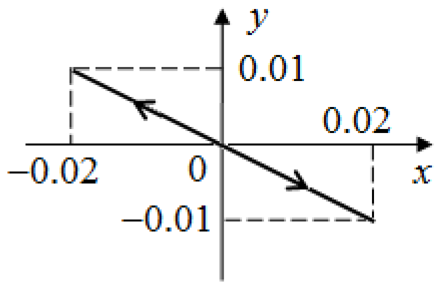
$$y = 0.01\cos\left(\pi t + \frac{\pi}{2}\right) = -0.01\sin\pi t.$$

The combined motion of the object is described by

$$\begin{cases} x = 0.02\sin\pi t, \\ y = -0.01\sin\pi t. \end{cases}$$

Dividing the second equation by the first equation we obtain

$$\frac{y}{x} = -\frac{1}{2}.$$



The equation of the trajectory is $y = -\dfrac{1}{2}x$. The displacement x is changing from -0.02 m to 0.02 m.

Problem 3.16

The motion of a 10 *g-particle is given by* $x = 5\sin\left(\frac{\pi}{5}t + \frac{\pi}{4}\right)$ *(cm). Find the maximum force that acts on the particle, and its total energy.*

Solution

Comparison of the general SHM equation $x = A\sin\left(\omega_0 t + \alpha\right)$ with the equation of particle motion $x = 0.05\sin\left(\frac{\pi}{5}t + \frac{\pi}{4}\right)$ (m) gives that the amplitude $A = 5\cdot10^{-2}$ m, the angular speed $\omega_0 = \pi/5$, and the phase constant (the initial phase of oscillations) $\alpha = \pi/4$.

The acceleration of the particle is

$$a = \ddot{x} = -A\omega_0^2 \sin(\omega_0 t + \alpha).$$

Therefore, according to the Second Newton's Law, the force that acts on the particle is

$$F = ma = -m\omega_0^2 A\sin(\omega_0 t + \alpha).$$

The maximum force makes

$$F_{\max} = m\omega_0^2 A.$$

The total energy of the particle is sum of its kinetic and potential energies

$$W = W_k + W_p,$$

where

$$W_k = \frac{mA^2\omega_0^2}{2}\cos^2(\omega_0 t + \alpha),$$

$$W_p = \frac{mA^2\omega_0^2}{2}\sin^2(\omega_0 t + \alpha).$$

The total energy is

$$W = \frac{mA^2\omega_0^2}{2}\left[\cos^2(\omega_0 t + \alpha) + \sin^2(\omega_0 t + \alpha)\right] = \frac{mA^2\omega_0^2}{2}.$$

Substituting the values, we obtain

$$F_{\max} = m\omega_0^2 A = 10^{-2} \cdot (\pi / 5)^2 \cdot 5 \cdot 10^{-2} = 2 \cdot 10^{-4} \text{ N}.$$

$$W = \frac{mA^2\omega_0^2}{2} = \frac{10^{-2} \cdot \pi^2 \cdot (5 \cdot 10^{-2})^2}{2 \cdot 25} = 4.9 \cdot 10^{-6} \text{ J}.$$

Problem 3.17

A 4.5-*kg object oscillates on a horizontal spring with amplitude of* 3.8 *cm. Its maximum acceleration is* 26 *m/s². Find the force constant* k*, the frequency, and the period of the oscillations.*

Solution

The spring constant is $k = \omega_0^2 m$. On other side, the maximum acceleration is $a_{\max} = A\omega_0^2$. Combining these two equations we obtain

$$k = \frac{ma_{\max}}{A} = \frac{4.5 \cdot 26}{0.038} = 3079 \text{ N/m}.$$

$$\upsilon_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{a_{\max}}{A}} = \frac{1}{2\pi}\sqrt{\frac{26}{0.038}} = 4.16 \text{ Hz}.$$

The period of oscillations is

$$T = \frac{1}{\upsilon_0} = \frac{1}{4.16} = 0.24 \text{ s}.$$

Problem 3.18

Determine the ratio of the kinetic energy W_k *of material point participating in SHM, to its potential energy* W_p *if the phase of oscillations is known.*

Solution

The displacement of a point is

$$x = A\cos(\omega_0 t + \alpha),$$

and its velocity is

$$v = \frac{dx}{dt} = -A\omega_0 \sin(\omega_0 t + \alpha)\,.$$

The kinetic and potential energies of the point are

$$W_k = \frac{mv^2}{2} = \frac{mA^2\omega_0^2}{2}\sin^2(\omega_0 t + \alpha)\,,$$

$$W_p = -\int_0^x F dx = -\int_0^x m\omega_0^2 x dx = \frac{mA^2x^2}{2} = \frac{mA^2\omega_0^2}{2}\cos^2(\omega_0 t + \alpha)\,.$$

The ratio is

$$\frac{W_k}{W_p} = \frac{\sin^2(\omega_0 t + \alpha)}{\cos^2(\omega_0 t + \alpha)} = \tan^2(\omega_0 t + \alpha)\,.$$

Problem 3.19

A *particle executes simple harmonic motion with amplitude of* 10 *cm. At what distance from the mean position are the kinetic and potential energies equal?*

Solution

Let $x = A\sin\left(\omega_0 t + \alpha\right)$ be the distance from the mean position where kinetic and potential energies are equal. The kinetic energy at this instant of time is

$$W_k = \frac{mv^2}{2} = \frac{mA^2\omega_0^2}{2}\cos^2\left(\omega_0 t + \alpha\right),$$

$$W_p = \frac{kx^2}{2} = \frac{mA^2\omega_0^2}{2}\sin^2(\omega_0 t + \alpha)\,.$$

Given that $W_k = W_p$, then

$$\frac{mA^2\omega_0^2}{2}\cos^2\left(\omega_0 t + \alpha\right) = \frac{mA^2\omega_0^2}{2}\sin^2(\omega_0 t + \alpha),$$

$$\cos^2\left(\omega_0 t + \alpha\right) = \sin^2(\omega_0 t + \alpha),$$

$$\tan^2(\omega_0 t + \alpha) = 1,$$

$$\omega_0 t + \alpha = \pm\frac{\pi}{4},$$

$$x = A\sin(\omega_0\, t + \alpha) = 0.1 \cdot \sin\frac{\pi}{4} = \pm\frac{0.1}{\sqrt{2}} = 0.071 \text{ m}.$$

Problem 3.20

A simple pendulum has time period $T = 4$ *s. How the length should be changed so the pendulum may complete* 15 *oscillations in* 30 *seconds?*

Solution

The initial period of the pendulum is

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

Its length is

$$l = \frac{T^2 g}{4\pi^2} = \frac{16 \cdot 9.8}{4\pi^2} = 4 \text{ m}.$$

New period according to the given data is to be $T_1 = \frac{t}{N} = \frac{30}{15} = 2$ s, and new length is

$$l_1 = \frac{T_1^2 g}{4\pi^2} = \frac{4 \cdot 9.8}{4\pi^2} = 1 \text{ m}.$$

The length change is

$$\Delta l = l_1 - l = 4 - 1 = 3 \text{ m}.$$

Problem 3.21

A simple pendulum of length l *is suspended from the ceiling of an elevator. Find the time period of oscillations if the elevator a) is going up with acceleration* a*; b) is going down with acceleration* a*; and c) is moving with uniform velocity.*

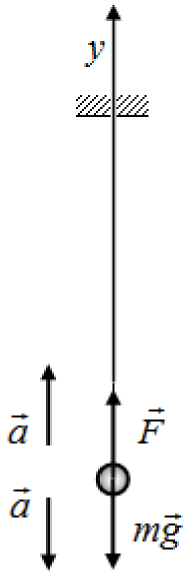
Solution

If $m\vec{g}$ is gravity, and $\vec{F}$ is tension, the equation of the bob motion according to Newton's 2nd Law is

$$m\vec{a} = m\vec{g} + \vec{F},$$

$$\vec{F} = m(\vec{a} - \vec{g}).$$

a) For the upward motion the tension and period of oscillations are



$$F = m(a - (-g)) = m(a + g),$$

$$T = 2\pi\sqrt{\frac{l}{g+a}}.$$

b) For downward motion

$$F = m(-a - (-g)) = m(g - a),$$

$$T = 2\pi\sqrt{\frac{l}{g-a}}.$$

c) When elevator is moving at uniform velocity its acceleration is zero and

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

Problem 3.22

A simple pendulum fixed in a car has a time period of 4 *seconds when the car is moving with a uniform velocity on a horizontal road. When the accelerator is pressed, the time period changes to* 3.99 *seconds. Find the acceleration of the car.*

Solution

When car is moving with uniform velocity the period of oscillations of the pendulum is

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

When car accelerates by a_x in horizontal direction the total acceleration $\vec{a}$ is vector some of horizontal a_x and vertical acceleration due to gravity $\vec{g}$; $\vec{a} = \vec{a}_x + \vec{g}$. Since the magnitude of total acceleration is $a = \sqrt{a_x^2 + g^2}$, the period is

$$T' = 2\pi\sqrt{\frac{l}{\sqrt{a_x^2 + g^2}}} = 2\pi\sqrt{\frac{g}{a}},$$

$$\frac{T}{T'} = \sqrt{\frac{a}{g}},$$

$$a = g\left(\frac{T}{T'}\right)^2 = 9.8\cdot\left(\frac{4}{3.99}\right)^2 = 9.85\,\text{m/s}^2,$$

$$a^2 = g^2 + a_x^2,$$

$$a_x = \sqrt{a^2 - g^2} = \sqrt{9.85^2 - 9.8^2} = 0.983\ \text{m/s}^2.$$

Problem 3.23

A mass 8 *g is attached to a horizontal spring that requires a force of* 0.01 *N to extend it to a length* 5 *cm greater than its natural length. What are the period, the frequency and the angular frequency of the simple harmonic motion of such a system?*

Solution

The length that the spring stretches is directly proportional to the applied force. The magnitude of this force is

$$F = kx,$$

where k is the force constant.

Proceeding from the given data, the force constant is

$$k = \frac{F}{x} = \frac{0.01}{0.05} = 0.2\ \text{N/m}.$$

The period, the frequency, and the angular frequency are, respectively

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{0.008}{0.2}} = 1.26\ \text{s},$$

$$\upsilon_0 = \frac{1}{T} = \frac{1}{1.26} = 0.8\,\text{Hz},$$

$$\omega_0 = 2\pi\upsilon_0 = 2\pi \cdot 0.8 = 1.6\pi = 5.03 \text{ rad/s}.$$

Problem 3.24

A block suspended from a vertical spring is in equilibrium. Show that the extension of the spring equals the length of an equivalent simple pendulum, i. e., a pendulum having frequency same as that of the block.

Solution

On the assumption of the same angular frequency of two pendulums

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{l}{g}},$$

where k is the spring constant, m is the mass of the block, and l is the length of the pendulum.

Therefore,

$$\frac{k}{m} = \frac{l}{g},$$

$$l = \frac{kg}{m}.$$

At equilibrium position of spring pendulum the gravity is equal to the elastic force

$$kx = mg,$$

where x is the extension of the spring.

$$x = \frac{mg}{k}.$$

Comparing the obtained expressions for the length of the simple pendulum l and the extension of the spring x we find that they are equal

$$x = \frac{mg}{k} = l.$$

Problem 3.25

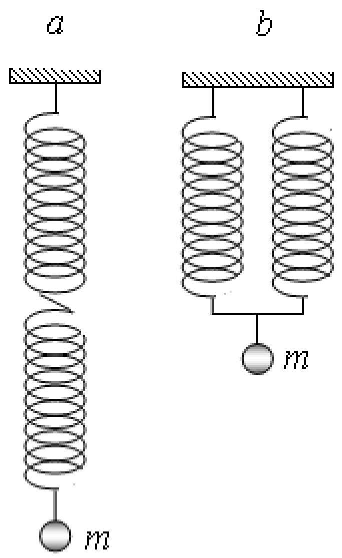
Find the periods of vertical oscillations of the block suspended with the help of two equal springs if the block is connected to springs a) in series, or b) in parallel.

Solution

In the equilibrium point, the force acted on the load is $|F| = kx$, and according to the Newton's 1st Law $mg = kx$, the spring extension is

$$x = \frac{mg}{k}.$$

a) When two springs are attached one at the end of the other, their extensions are equal, and the total extension is $x_1 = 2x = \frac{2mg}{k}$.



On the other side, $x_1 = \frac{mg}{k_1}$.

Equating these two expressions, we obtain

$$\frac{2mg}{k} = \frac{mg}{k_1}.$$

$$k_1 = \frac{k}{2}.$$

b) If the block is connected to the springs in parallel, the total extension is

$$x_2 = x = \frac{(m/2)g}{k} = \frac{mg}{2k}.$$

Again, $x_2 = \frac{mg}{k_2}$.

As a result,

$$k_2 = 2k .$$

The periods of oscillations for two examined cases are

$$T_1 = 2\pi\sqrt{\frac{m}{k_1}} \qquad \text{and} \qquad T_2 = 2\pi\sqrt{\frac{m}{k_2}},$$

and their ratio is

$$\frac{T_1}{T_2} = \sqrt{\frac{k_2}{k_1}} = \sqrt{\frac{2k \cdot 2}{k}} = \sqrt{4} = 2\,.$$

Problem 3.26

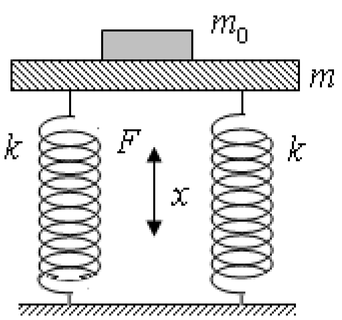
A tray of mass m = 12 *kg is supported by two identical springs as shown in figure. When the tray is pressed down slightly and then released, it executes SHM with a time period of* 1.5 *s. a) What is the spring constant of each spring? b) When the block of mass* m_0 *is placed on the tray, the period of SHM changes to* 3 *s, what is the mass of the block?*

Solution

a) Let $m = 12\,\text{kg}$ be the mass of tray and k is the force constant of each spring. When the tray is pressed down slightly it begins to execute SHM. Let x be the downward displacement of the tray at any time t, then each spring exerts a restoring force kx upward.

Net restoring force on tray is

$$F = -kx - kx = -2kx\,.$$



Clearly, the effective force constant of two springs is $k_0 = 2k$. Time period is

$$T = 2\pi\sqrt{\frac{m}{k_0}} = 2\pi\sqrt{\frac{m}{2k}}\,,$$

$$k = \frac{2\pi^2 m}{T^2}\,.$$

Given $m = 12\,\text{kg}$, $T = 1.5\,\text{s}$,

$$k = \frac{2\pi^2 \cdot 12}{1.5^2} = 105.2 \text{ N/m}.$$

b) When a block of mass m_0 is placed on the tray, net mass is $m + m_0$. New time period is

$$T_1 = 2\pi\sqrt{\frac{m + m_0}{2k}}\,,$$

$$m+m_0=\frac{T_0^2\cdot 2k}{4\pi^2}=\frac{T_0^2\cdot k}{2\pi^2}=\frac{3^2\cdot 105.2}{2\pi^2}=48\,\text{kg},$$

$$m_0=48-m=48-12=36\ \text{kg}.$$

Problem 3.27

A rod of the length $l=1$ m oscillates about the axis passing through its end. a) Find the period of oscillations of the rod; b) Determine the location of the pivot point that provides the maximum frequency of oscillations.

Solution

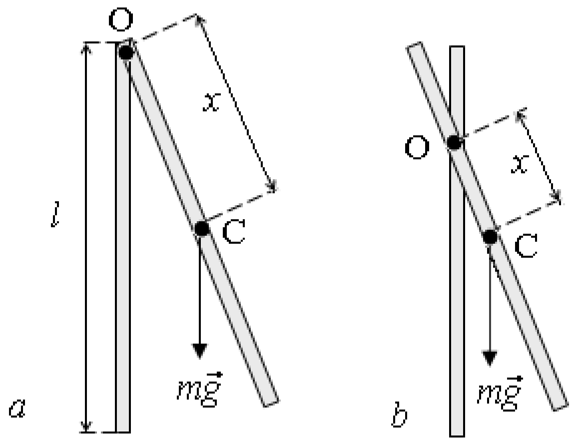
a) The rod is a physical (compound) pendulum, and its period of oscillations is determined by $T=2\pi\sqrt{\dfrac{I}{mgx}}$, where $x=l/2$ is the distance between the centre of mass and the pivot point, and I is the moment of inertia respectively the axis passing through the pivot point. This moment of inertia according to Steiner theorem is

$$I=I_0+mx^2=\frac{ml^2}{12}+m\left(\frac{l}{2}\right)^2=\frac{ml^2}{3}.$$

$$T=2\pi\sqrt{\frac{I}{mgx}}=2\pi\sqrt{\frac{ml^2}{3\cdot mg(l/2)}}=2\pi\sqrt{\frac{2l}{3g}}=1.63\ \text{s}.$$

b) The angular frequency of the compound pendulum is

$$\omega_0=\frac{2\pi}{T}=\sqrt{\frac{mgx}{I}}.$$



The moment of inertia respectively new pivot point according to Steiner theorem is

$$I=\frac{ml^2}{12}+mx^2,$$

where x is required value.

After that

$$\omega_0 = \sqrt{\frac{mgx}{\frac{ml^2}{12} + mx^2}} = \left(\frac{12gx}{l^2 + 12x^2} \right)^{1/2}.$$

Maximization of the function $\omega_0(x)$ allows finding the required distance x between the pivot point and centre of mass. For this purpose we have to take the derivate of $\omega_0(x)$ and equate it to zero.

$$\frac{d\omega_0}{dx} = \frac{1}{2}\left(\frac{12gx}{l^2 + 12x^2} \right)^{-\frac{1}{2}} \left[\frac{12g\left(l^2 + 12x^2\right) - 12gx(12 \cdot 2x)}{\left(l^2 + 12x^2\right)^2} \right] = \frac{\sqrt{3g}(l^2 - 12x^2)}{\sqrt{x(l^2 + 12x^2)^3}} = 0,$$

$$l^2 - 12x^2 = 0,$$

$$x = \frac{l}{\sqrt{12}} = \frac{1}{2\sqrt{3}} = 0.29 \text{ m}.$$

Problem 3.28

A mass of 2 *kg oscillates on a spring with constant* 50 *N/m. By what factor does the frequency of oscillation decrease when a damping force with constant* r = 12 *(coefficient of resistance of the medium) is introduced?*

Solution

The natural angular frequency of oscillation (without resistance) is given by

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{50}{2}} = 5 \text{ rad/s}.$$

The damping coefficient is

$$\beta = \frac{r}{2m} = \frac{12}{2 \cdot 2} = 3 \text{ s}^{-1}.$$

The frequency of damped oscillation is given by

$$\omega = \sqrt{\omega_0^2 - \beta^2} = \sqrt{5^2 - 3^2} = 4 \text{ rad/s}.$$

Thus the frequency decreases by 1 rad/s, or by 20 percent of its original value.

Problem 3.29

Amplitude of damped oscillations of the simple pendulum decreased twice during $t_1 = 1$ *min. By what factor does the amplitude decreased during* $t_2 = 3$ *min?*

Solution

Amplitude of the damped oscillations depends on the time according to $A_t = A_0 e^{-\beta t}$, where A_0 is initial amplitude. Therefore,

$$\frac{A_0}{A_{t_1}} = \frac{\cancel{A_0}}{\cancel{A_0} \cdot e^{-\beta t_1}} = e^{\beta t_1} = 2,$$

$$\beta t_1 = \ln 2 .$$

The damping coefficient is $\beta = \dfrac{\ln 2}{t_1}$.

For the second time interval

$$\frac{A_0}{A_{t_2}} = \frac{\cancel{A_0}}{\cancel{A_0} \cdot e^{-\beta t_2}} = e^{\beta t_2} = e^{\frac{\ln 2 \cdot t_2}{t_1}} = e^{\frac{0.693 \cdot 3}{1}} = 8 .$$

Thus the amplitude of damped oscillations is decreased by factor of 8 during 3 minutes.

Problem 3.30

Amplitude of simple pendulum oscillations of the length $l = 1$ *m during* 10 *min was decreased twice. Determine the damping coefficient* β, *logarithmic decrement* δ *and the number of oscillations* N *during this time interval. Find the equation of oscillations if initially the pendulum was pulled sideways to a distance of* 5 *cm and released.*

Solution

The amplitude of damped oscillations was decreased twice during $t = 10$ min = 600 s.

Then the ratio of the initial amplitude A_0 and the amplitude after time t - A_t is:

$$\frac{A_0}{A_t} = \frac{A_0}{A_0 e^{-\beta t}} = e^{\beta t} = 2,$$

$$\beta t = \ln 2,$$

$$\beta = \frac{\ln 2}{t} = \frac{0.693}{600} = 10^{-3}\ \text{s}^{-1}.$$

For finding the logarithmic decrement we have to know the period of damped oscillations T. Firstly, find the period and angular frequency of simple harmonic motion (without damping).

$$T_0 = 2\pi\sqrt{\frac{l}{g}} = 2\ \text{s},$$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi\ \text{rad/s}.$$

The angular frequency of damped oscillations is

$$\omega = \sqrt{\omega_0^2 - \beta^2} = \sqrt{\pi^2 - 10^{-6}} \approx \omega_0.$$

Since the angular frequency ω of damped oscillations is almost equals the own angular frequency ω_0, the period of damped oscillations is $T \approx T_0 = 2$ s. The logarithmic decrement is

$$\delta = \beta T = 2\cdot 10^{-3}.$$

The number of oscillations N for time t may be found from

$$\beta t = \beta N T = \ln 2.$$

$$N = \frac{\ln 2}{\beta T} = \frac{0.693}{2\cdot 10^{-3}} = 346.6.$$

The distance by which the pendulum was deviated at $t = 0$ is its initial amplitude $A_0 = 5\cdot 10^{-2}$m. Since $x(0) = A_0$, we use the cosine function with initial phase equaled to zero for the equation of examined damped oscillations.

$$x = 5\cdot 10^{-2} e^{-0.001\cdot t}\cos\pi t\ \text{(m)}.$$

Problem 3.31

Harmonic oscillator of the mass $m = 0.25$ kg moves attached to the spring with spring constant $k = 85$ N/m in the medium with resistance coefficient $r = 0.07$ kg/s. Calculate a) the period of its oscillation; b) the number of oscillations in which its amplitude will become half of its original value; c) the number of oscillations in which its mechanical energy will drop to one-half of its initial value; and d) the quality factor.

Solution

a) The damping coefficient β of the oscillating system is

$$\beta = \frac{r}{2m} = \frac{0.07}{2 \cdot 0.25} = 0.14 \ \text{s}^{-1}.$$

The natural angular frequency of the oscillator (without friction) is

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{85}{0.25}} = 18.44 \ \text{rad/s}.$$

The angular frequency of damped oscillator is

$$\omega = \sqrt{\omega_0^2 - \beta^2} = \sqrt{18.44^2 - 0.14^2} \approx 18.44 \ \text{rad/s}.$$

As it is seen, $\omega = \omega_0$ due to $\omega_0 \gg \beta$.
Then the period of oscillation is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{18.44} = 0.34 \ \text{s}.$$

b) The ratio of the initial amplitude and the amplitude after N oscillations is

$$\frac{A_0}{A_t} = \frac{A_0}{A_0 \cdot e^{-\beta t}} = e^{\beta t} = e^{\beta NT} = 2 .$$

On taking natural logarithm of both sides and rearranging terms, we get

$$\beta NT = \ln 2 ,$$

$$N = \frac{\ln 2}{\beta T} = \frac{0.693}{0.14 \cdot 0.34} = 14.56 .$$

c) The ratio of the initial energy and the energy after N_1 oscillations is

$$\frac{W_0}{W} = \frac{W_0}{W_0 \cdot e^{-2\beta t}} = e^{2\beta t} = e^{2\beta N_1 T} = 2,$$

$$2\beta N_1 T = \ln 2,$$

$$N_1 = \frac{\ln 2}{2\beta T} = \frac{0.693}{2 \cdot 0.14 \cdot 0.34} = 7.28.$$

d) The quality factor is

$$Q = \frac{\pi}{\delta} = \frac{\pi}{\beta T} = \frac{\pi}{0.14 \cdot 0.34} = 66.$$

Problem 3.32

A damped oscillator loses 3.5% *of its energy during each cycle. What is its Q factor? How much cycles elapse before half of its original energy is dissipated?*

Solution

The total energy of oscillator depends on the square of amplitude

$$W = \frac{m\omega_0^{\,2} A^2}{2}.$$

Therefore, the energy of damped oscillator is

$$W = \frac{m\omega^2 A^2}{2} = \frac{m\omega^2 A_0^2}{2} \cdot e^{-2\beta t} = W_0 \cdot e^{-2\beta t}.$$

The damped oscillator loses 3.5% of its energy during one cycle. This implies that after the time $t = T$ the energy of oscillator equals to

$$100\% - 3.5\% = 96.5\%.$$

Thus,

$$\frac{W_0}{W_T} = \frac{\cancel{W_0}}{\cancel{W_0} \cdot e^{-2\beta T}} = e^{2\beta T} = \frac{100}{96.5} = 1.036,$$

$$2\beta T = \ln 1.036 = 0.0356,$$

$$\beta T = \delta = \frac{0.0356}{2} = 1.78 \cdot 10^{-2},$$

$$Q = \frac{\pi}{\delta} = \frac{\pi}{1.78 \cdot 10^{-2}} = 176.$$

The oscillator losses the half of its energy during time t.

$$\frac{W_0}{W_t} = \frac{\cancel{W_0}}{\cancel{W_0} \cdot e^{-2\beta t}} = e^{2\beta t} = 2,$$

$$2\beta t = \ln 2 = 0.693,$$

$$\beta t = \frac{0.693}{2} = 0.3465.$$

On the other hand,

$$\beta t = \beta N T = N\delta.$$

Then,

$$N\delta = 0.3465,$$

$$N = \frac{0.3465}{\delta} = \frac{0.3465}{1.78 \cdot 10^{-2}} = 19.5 \approx 20.$$

Problem 3.33

An oscillator with a period of 1 *s has amplitude that decreases by* 1% *during each complete oscillation. a) If the initial amplitude is* 10.2 *cm, what will be the amplitude after* 35 *oscillations? b) At what time will the energy be reduced to* 46% *of its initial value?*

Solution

a) The ratio of initial amplitude and the amplitude after the period is

$$\frac{A_0}{A_T} = \frac{A_0}{A_0 \cdot e^{-\beta T}} = e^{\beta T} = \frac{100}{99} = 1.01.$$

It means that the logarithmic decrement is equal to

$$\beta T = \delta = \ln 1.01 = 0.01.$$

The damping coefficient is

$$\beta = \frac{0.01}{T} = \frac{0.01}{1} = 0.01\ \text{s}^{-1}.$$

If the time for amplitude decrease is t, the ratio of amplitudes is

$$\frac{A_0}{A_t} = \frac{A_0}{A_0 \cdot e^{-\beta t}} = e^{\beta t} = e^{\beta NT} = e^{N\delta},$$

$$A_t = \frac{A_0}{e^{N\delta}} = \frac{10.2}{e^{35\cdot 0.01}} = 7.18\ \text{cm}.$$

b) The ratio of initial energy and the energy after time t_1 is

$$\frac{W_0}{W_{t_1}} = \left(\frac{A_0}{A_{t_1}}\right)^2 = \frac{A_0^2}{A_0^2 \cdot e^{-2\cdot\beta\cdot t_1}} = e^{2\cdot\beta\cdot t_1},$$

$$\frac{W_0}{W_{t_1}} = \frac{W_0}{0.46 \cdot W_0} = \frac{1}{0.46} = 2.17,$$

$$e^{2\cdot\beta\cdot t_1} = 2.17,$$

$$2\beta t_1 = \ln 2.17$$

$$t_1 = \frac{\ln 2.17}{2\beta} = 38.74\ \text{s}.$$

Problem 3.34

A compound pendulum with equivalent length of 24.7 *cm executes damped oscillations. In what time will the energy of the oscillation become* 10% *of the initial energy if the logarithmic decrement factor is* 0.01?

Solution

The ratio of initial energy and energy after the time interval t is

$$\frac{W_0}{W_t} = \frac{W_0}{W_0 \cdot e^{-2\beta t}} = e^{2\beta t}.$$

On other hand, according to the given data

$$\frac{W_0}{W_t} = \frac{W_0}{0.1 \cdot W_0} = 10.$$

$$e^{2\beta t} = 10.$$

$$2\beta t = \ln 10.$$

$$t = \frac{\ln 10}{2\beta}.$$

The natural angular velocity of compound pendulum is

$$\omega_0 = \sqrt{\frac{g}{L}} = \sqrt{\frac{9.8}{0.247}} = 6.3 \text{ rad/s}.$$

Since the logarithmic decrement is $\delta = 0.01$ the given oscillator is the system with weak damping. Therefore, the angular frequency of damped oscillations may be taken as equaled to the natural angular frequency, $\omega_0 = \omega$. Moreover, $\delta = \beta T = \frac{\beta \cdot 2\pi}{\omega}$, hence, $\beta = \frac{\delta\omega}{2\pi}$. As a result,

$$t = \frac{\ln 10}{2\beta} = \frac{\ln 10 \cdot 2\pi}{2\delta\omega} = \frac{\ln 10 \cdot \pi}{\delta\omega} = \frac{2.3 \cdot \pi}{0.01 \cdot 6.3} = 115 \text{ s}.$$

Problem 3.35

A 145-*MHz radio signal propagates along a cable. Measurement shows that the wave crests are spaced* 1.25 *m apart. What is the speed of the waves on the cable? Compare with the speed of light in vacuum.*

Solution

The distance between adjacent wave crests is equal to the wavelength $\lambda = \frac{v}{\upsilon}$, so the wave speed in the cable is

$$v = \lambda \cdot \upsilon = 1.25 \cdot 145 \cdot 10^6 = 1.81 \cdot 10^8 \text{ m/s}.$$

Since the speed of light in vacuum is $c = 3 \cdot 10^8$ m/s,

$$\frac{v}{c} = \frac{1.81 \cdot 10^8}{3 \cdot 10^8} = 0.6\,.$$

Problem 3.36

Ultrasound used in a particular medical imager has frequency 4.8 *MHz and wavelength* 0.31 *mm. Find the angular frequency, the wave number and the wave speed.*

Solution

The angular frequency related to the frequency as $\omega = 2\pi\upsilon$, therefore,

$$\omega = 2\pi \cdot 4.8 \cdot 10^6 = 3.02 \cdot 10^7 \text{ rad/s.}$$

The wave number is

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{0.31 \cdot 10^{-3}} = 2.03 \cdot 10^4 \text{ m}^{-1}.$$

The speed of the ultrasound wave is

$$v = \lambda\upsilon = 0.31 \cdot 10^{-3} \cdot 4.8 \cdot 10^6 = 1.49 \cdot 10^3 \text{ m/s.}$$

Problem 3.37

Write the expression for a harmonic wave that has a wavelength of 2.8 *m and propagates with a speed of* 13.3 *m/s. The amplitude of the wave is* 0.12 *m, initial phase is zero. Estimate two cases: a)* $\xi(0,0) = 0$*; b)* $\xi(0,0) = A$.

Solution

The general expressions describing the propagating wave in accordance with initial conditions are

$$\xi(x,t) = A\sin\left(\frac{2\pi}{T}t - \frac{2\pi}{\lambda}x\right) \text{ for } \xi(0,0) = 0,$$

$$\xi(x,t) = A\cos\left(\frac{2\pi}{T}t - \frac{2\pi}{\lambda}x\right) \text{ for } \xi(0,0) = A\,.$$

The wavelength is $\lambda = vT$, therefore, the period of oscillations is

$$T = \frac{\lambda}{v} = \frac{2.8}{13.3} = 0.21 \text{ s.}$$

Depending on the initial conditions the equations of the wave are

a) $\xi(x,t) = 0.12\sin\left(\frac{2\pi}{0.21}t - \frac{2\pi}{2.8}x\right)$,

b) $\xi(x,t) = 0.12\cos\left(\frac{2\pi}{0.21}t - \frac{2\pi}{2.8}x\right)$.

Problem 3.38

A transverse wave propagates along a stretched string with the velocity 15 *m/s. A period of oscillations of the points of the string is* 1.2 *s, and amplitude is* 2 *m. Find the phase, displacement, velocity and acceleration of the point* 45 *m distant from the vibration source on the instant of time* t = 4 *s. Determine the maximum velocity and the maximum acceleration of the point. The initial phase is zero,* $\xi(0,0) = A$.

Solution

The wavelength is $\lambda = vT = 15 \cdot 1.2 = 18$ m.

The displacement of the point may be determined using the equation of the travelling wave where the initial phase equals to zero α=0

$$\xi(x,t) = A\cos\left(\frac{2\pi}{T}t - \frac{2\pi}{\lambda}x + \alpha\right),$$

$$\xi(45,4) = 2\cos\left(\frac{2\pi}{1.2}\cdot 4 - \frac{2\pi}{18}\cdot 45\right) = 1 \text{ m.}$$

The phase for the examined point of the wave is

$$\frac{2\pi}{1.2}\cdot 4 - \frac{2\pi}{18}\cdot 45 = \frac{5\pi}{3} = 5.23 \text{ rad.}$$

The speed and acceleration of the point may be determined by time differentiating of $\xi(x,t)$ and $\dot{\xi}(x,t)$, respectively:

$$\dot{\xi}(x,t)=\frac{\partial\xi}{\partial t}=-A\cdot\frac{2\pi}{T}\sin\left(\frac{2\pi}{T}t-\frac{2\pi}{\lambda}x\right)=-2\cdot\frac{2\pi}{1.2}\cdot\sin\frac{5\pi}{3}=9.07\ \text{m/s}.$$

$$\ddot{\xi}(x,t)=\frac{\partial^2\xi}{\partial t^2}=-A\cdot\left(\frac{2\pi}{T}\right)^2\cos\left(\frac{2\pi}{T}t-\frac{2\pi}{\lambda}x\right)=-2\cdot\left(\frac{2\pi}{1.2}\right)^2\cos\frac{5\pi}{3}=-27.4\ \text{m/s}^2.$$

The maximum speed may be calculated from the first equation

$$A\cdot\frac{2\pi}{T}=2\cdot\frac{2\pi}{1.2}=10.5\ \text{m/s}.$$

The second equation gives the maximum acceleration of the point

$$A\cdot\left(\frac{2\pi}{T}\right)^2=2\cdot\left(\frac{2\pi}{1.2}\right)^2=54.8\ \text{m/s}^2.$$

Problem 3.39

Find the wavelength and the phase difference of two oscillating points distant by 10 *and* 16 *m from the vibrating source, respectively. The period is* 0.04 *s, the velocity of the wave propagation is* 300 *m/s.*

Solution

If the period of oscillations is $T=0.04$ s, and the speed of the wave is $v=300$ m/s, the wavelength equals to

$$\lambda=vT=300\cdot 0.04=12\ \text{m}.$$

The equations of the oscillations of the points with the coordinates $x_1=10$ m and $x_2=16$ m at the travelling wave propagation are

$$\xi_1(x_1,t)=A\cos(\omega t-kx_1),$$

$$\xi_2(x_2,t)=A\cos(\omega t-kx_2).$$

The phase difference for these points is

$$\Delta\varphi=|\varphi_1-\varphi_2|=(\omega t-kx_1)-(\omega t-kx_2)=k(x_2-x_1),$$

where $k=\dfrac{2\pi}{\lambda}=\dfrac{2\pi}{vT}$ is the wave number.

Finally, the phase difference is

$$\Delta\varphi = \frac{2\pi}{vT}\left(x_2 - x_1\right) = \frac{2\pi}{300\cdot 0.04}(16-10) = \pi\,.$$

It means that the given oscillations are opposite in phase.

Problem 3.40

Find the frequency of the sound wave in the tube of the length L $=1$ *m, if its both ends are (a) open; (b) closed, (c) one end is opened and another end is closed. The speed of sound* v *= 340 m/s.*

Solution

The standing waves are created in all cases of the tubes. The open and closed ends reflect waves differently. The closed end of the tube is the node, and the open end is an antinode. The longest standing wave in a tube of length L with two open ends has displacement antinodes at the both ends and only one node between them. The frequency in this case is so called fundamental frequency.

$$\lambda = 2L\,,$$

$$\upsilon_1 = \frac{v}{\lambda} = \frac{v}{2L} = \frac{340}{2\cdot 1} = 170 \text{ Hz.}$$

The next standing wave in this tube is second harmonic. It also has displacement antinodes at each end with two nodes between them.

$$\lambda = L\,.$$

The frequency is equal to

$$\upsilon_2 = \frac{v}{\lambda} = \frac{v}{L} = \frac{340}{1} = 340 \text{ Hz.}$$

An integer number of half wavelength have to fit into the tube of length L $L = n\frac{\lambda}{2}$, the wavelength will be $\lambda = \frac{2L}{n}$, and the frequency $\upsilon_n = n\frac{v}{2L}$ (natural frequencies, or harmonics).

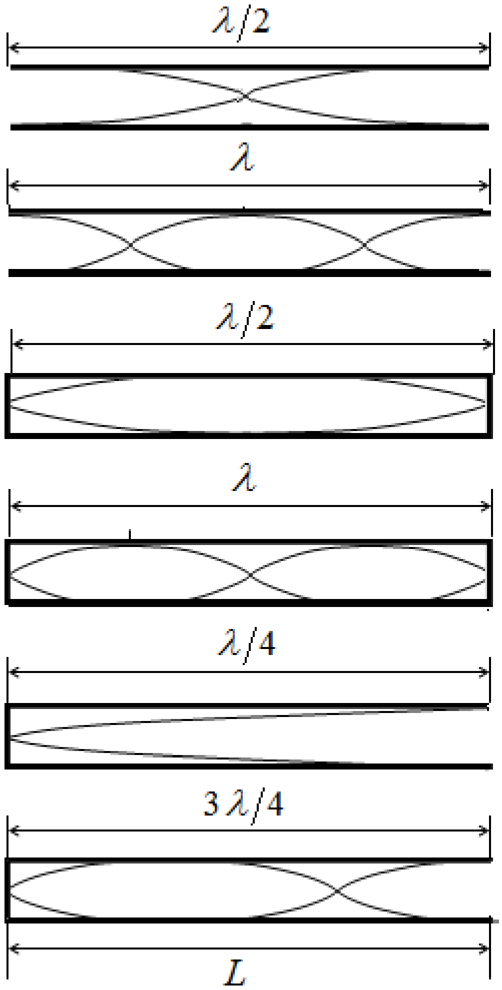
For a tube with two closed ends the longest standing wave is $\lambda = 2L$: two nodes at the closed ends and one antinode between them. The fundamental frequency is

$$\upsilon_1 = \frac{v}{\lambda} = \frac{v}{2L} = \frac{340}{2 \cdot 1} = 170 \text{ Hz.}$$

The second natural frequency (at $\lambda = L$) is

$$\upsilon_2 = \frac{v}{\lambda} = \frac{v}{L} = \frac{340}{1} = 340 \text{ Hz.}$$

The longest standing wave in a tube of length L with one open end and one closed end has a displacement antinode at the open end and a displacement node at the closed end. This is the fundamental frequency.



$$\lambda = 4L,$$

$$\upsilon = \frac{v}{\lambda} = \frac{v}{4L} = \frac{340}{4 \cdot 1} = 85 \text{ Hz.}$$

Next frequency will be at $L = 3\lambda/4$,

$$\lambda = 4L/3,$$

$$\upsilon = \frac{v}{\lambda} = \frac{3v}{4L} = \frac{3 \cdot 340}{4 \cdot 1} = 255 \text{ Hz}$$

An odd-integer number of quarter wavelength have to fit into the tube of length L.

$$L = n\frac{\lambda}{4},$$

$$\lambda = \frac{4L}{n},$$

$$\upsilon_n = \frac{v}{\lambda} = n\frac{v}{4L},$$

where n is an odd number.

For a tube with one open end and one closed end all frequencies

$$\upsilon_n = \frac{v}{\lambda} = n\frac{v}{4L} = n\upsilon_1,$$

with n equaled to an odd integer are natural frequencies, i. e. only odd harmonics of the fundamental are natural frequencies.

Problem 3.41

A source of the sound is moving towards detector at rest with speed 50 *m/s. If the true frequency of sound emitted by the source is* 1100 *Hz, calculate the apparent frequency. Find the observed frequency when the source is moving away from the detector with same speed. Take the speed of sound in air* $v = 340$ *m/s.*

Solution

The phenomenon examined in this problem is Doppler effect. When both the source and the detector are in motion along the same straight line, the moving detector will receive a wave whose apparent frequency υ' depends on the actual frequency of the source υ' as

$$\upsilon' = \upsilon\frac{v \pm v_D}{v \pm v_S} \ ,$$

where v is the speed of sound through the air.

If the detector is at rest ($v_D = 0$) and the source is moving towards the detector, the observed frequency is

$$\upsilon' = \upsilon\frac{v}{v - v_S} = 1100 \cdot \frac{340}{340 - 50} = 1290 \text{ Hz}.$$

If the detector is at rest and the source is moving away from the detector, the observed frequency is

$$\upsilon' = \upsilon\frac{v}{v + v_S} = 1100 \cdot \frac{340}{340 + 50} = 959 \text{ Hz}.$$

As is evident from calculations when the source is approaching detector the observed frequency is more than the actual frequency, $\upsilon' > \upsilon$, and when it is moving away from detector the observed frequency is less than the actual frequency, $\upsilon' > \upsilon$.

Problem 3.42

A railway engine and a car are moving parallel but in opposite direction with velocities 28 m/s and 22m/s, respectively. The frequency of engine's whistle is 600 Hz and the velocity of sound is 340 m/s. Calculate the frequency of sound heard in the car when a) the car and engine are approaching each other, b) both are moving away from each other.

Solution

a) The observed frequency of sound when the engine and a car are approaching each other according to the expression for Doppler effect is

$$\upsilon' = \upsilon \frac{v + v_D}{v - v_S} = 600 \frac{340 + 22}{340 - 28} = 696 \text{ Hz}.$$

b) When the car and the engine are moving away from each other the frequency of the sound heard in the car is

$$\upsilon' = \upsilon \frac{v - v_D}{v + v_S} = 600 \frac{340 - 22}{340 + 28} = 518 \text{ Hz}.$$

CONTROL PROBLEMS

1. A particle which executes SHM along a straight line has its motion represented by $x = 4\sin\left(\frac{\pi t}{3} + \frac{\pi}{6}\right)$ (m). Find the amplitude; time period; frequency; initial phase; velocity and acceleration at $t = 1$ s. [$A = 4$ m; $T = 6$ s; $\upsilon = 0.167$ Hz; $\alpha = \pi/6$ rad; $v = 0$; $a = -13.16$ m/s^2]

2. A point is executing SHM with a period $T = 3.14$ s. When it is passing through the centre of its path, its velocity is 0.1 m/s. What is its velocity when it is at a distance of 0.03 m from the mean position? [0.08 m/s]

3. A particle moves with simple harmonic motion in a straight line. Its maximum speed is 4 m/s and its maximum acceleration is 16 m/s^2. Find a) the time period of the motion, b) the amplitude of the motion. [$\omega = 4$ rad/s, $A = 1$ m]

4. A particle makes SHM along a straight line and its velocities when passing through points 3 and 4 cm from the centre of its path are 16 and 12 cm/s, respectively. Find a) the amplitude; b) the time period of motion. [$A = 5$ cm; $T = 1.57$ s]

5. A particle performs SHM with a period of 16 s. At time $t = 2$ s, the particle passes through the origin; while at $t = 4$ s, its velocity is 4 m/s. Find the amplitude of the motion. [$A = 32\sqrt{2}/\pi$]

6. Write the equation of the resultant oscillation obtained by superposition of two identical harmonic oscillations of the same period $T = 10$ s and amplitude $A_1 = A_2 = A = 0.01$ m. The difference of phase between them is $\pi/4$ and the epoch of one of them is zero. [$x = 0.018\cos\left(\frac{\pi}{5}t + \frac{\pi}{8}\right)$ m]

7. The resultant oscillation of two identical harmonic oscillations directed in the same direction has the same period and amplitude as that of the component SHMs. Calculate the difference in their initial phases. [$\Delta\alpha = 2\pi/3$]

8. Calculate the amplitude and epoch of the resultant of two identical harmonic oscillations given by $x_1 = 4\sin\pi t$ (cm) and $x_2 = 3\sin(\pi t + \pi/2)$ (cm). [$A = 0.05$ m; $\alpha = \arctan 0.75 = 36.9° = 0.64$ rad]

9. Consider three simple harmonic motions of the same frequency acting in parallel on a particle, simultaneously. The amplitudes are 0.25, 0.2 and 0.1 cm. The phase difference between the first and the second is $\pi/4$ and between the second and the third is $\pi/6$. Find the resultant amplitude and the phase with respect to the first vibration motion. [$A = 0.48$ cm, $\alpha = \pi/6$]

10. A point participates simultaneously in two mutually perpendicular oscillations $x = \sin\pi t$ and $y = 2\sin\left(\pi t + \dfrac{\pi}{2}\right)$. Draw its trajectory. [$4x^2 + y^2 = 4$]

11. Determine the trajectory of the motion of a point which participated simultaneously in two mutually perpendicular oscillations given by (a) $x = 2\sin\omega t$, $y = 2\cos\omega t$; (b) $x = \sin\omega t$; $y = 4\sin(\omega t + \pi)$.[$x^2 + y^2 = 4$; $y = -4x$]

12. A point takes part in two harmonic oscillations at right angle: (a) $x = 0.5\sin(\omega t + \pi)$, $y = 2\cos(\omega t + \pi)$; (b) $x = 3\sin\left(\omega t + \dfrac{\pi}{2}\right)$, $y = 2\cos\omega t$. Find the trajectory of the point and the direction of its motion. [(a) $4x^2 + 0.25y^2 = 1$, clockwise; (b) $y = \dfrac{2}{3}x$]

13. A 2.4-kg object is attached to a horizontal spring of force constant $k = 4.5$ kN/m. The spring is stretched by 10 cm from equilibrium and released. Find a) the frequency of the motion, b) the period, c) the amplitude, d) the maximum speed, and e) the maximum acceleration. f) When does the object first reach its equilibrium position? What is its acceleration at this time? [$\upsilon = 6.89$ Hz; $T = 0.145$ s; $A = 0.1$ m; $v_{\max} = 4.33$ m/s; $a_{\max} = 187$ m/s²; $t = T/4 = 36.25$ ms; $a = 0$]

14. Consider a particle moving in a simple harmonic motion according to the equation $x = 2\cos(50\pi t + 0.205\pi)$, where x is in centimeter and t in second.

The motion is started at $t=0$. a) When does the particle come to rest for the first time? b) When does the acceleration have its maximum magnitude for the first time? c) When does the particle come to rest for the second time? [0.016 s, 0.016 s, 0.036 s]

15. A mass M is suspended from a spring of the negligible mass. The spring is pulled a little and then released so that the mass executed simple harmonic oscillations with a time period T. If the mass is increased by m then the time period becomes $5T/4$. Find the ratio of m/M. [9/16]

16. For the SHM of time period T calculate the time taken for the displacement to change value from half the amplitude to the amplitude [$t=T/6$]

17. A mass of 1 g vibrates through 1 mm on each side of the middle point of its path and makes 500 complete vibrations per second. Assuming its motion to be simple harmonic, show that the maximum force acting on the particle is equal to π^2 N.

18. A 100 g mass vibrates horizontally without friction at the end of the horizontal spring for which the spring constant is 10 N/m. The mass is displaced 0.5 cm from its equilibrium and released. Find: a) Its maximum speed, b) Its speed when it is 0.3 cm from equilibrium. c) What is its acceleration in each of these cases? [$v_{max}=0.05$ m/s; $|v|=0.04$ m/s; $a=-0.03$ m/s²]

19. A 3-kg object attached to a horizontal spring oscillates with an amplitude A = 10 cm and a frequency υ = 2.4 Hz. a) What is the force constant of the spring? b) What is the period of the motion? c) What is the maximum speed of the object? d) What is the maximum acceleration of the object? [$k=682$ N/m; $T=0.417$ s; $v_{max}=1.508$ m/s; $a_{max}=22.7$ m/s²]

20. A 0.4-kg block attached to a spring of force constant 12 N/m oscillates with an amplitude of 8 cm. Find a) the maximum speed of the block, b) the speed and acceleration of the block when it is at x = 4 cm from the equilibrium position, and c) the time it takes the block to move from $x=0$ to $x=4$ cm. [$v_{max}=0.438$ m/s; $v=0.379$ m/s; $a=1.2$ m/s², $\Delta t=0.0956$ s]

21. A mass M attached to a spring oscillates with a period of 2 s. If the mass is increased by 2 kg, the period increases by one second. Find the initial mass M assuming that Hooke's law is obeyed. [$M=1.6$ kg]

22. Find the total energy of a 3-kg object oscillating on a horizontal spring with an amplitude of 10 cm and a frequency of 2.4 Hz. [$W = 3.4$ J]

23. A 1.5-kg object oscillates with simple harmonic motion on a spring of force constant $k = 500$ N/m. Its maximum speed is 70 cm/s. a) What is the total energy? b) What is the amplitude of the oscillation? [$W = 0.368$ J; $A = 3.83$ cm]

24. An object oscillates on a spring with amplitude of 4.5 cm. Its total energy is 1.4 J. What is the force constant of the spring? [$k = 1383$ N/m]

25. In SHM if the displacement is one half of the amplitude, show that the kinetic energy and potential energy are in the ratio 3:1.

26. An object oscillates with amplitude of 5.8 cm on a horizontal spring of force constant 1.8 kN/m. Its maximum speed is 2.20 m/s. Find a) the mass of the object, b) the frequency of the motion, and c) the period of the motion. [$m = 1.25$ kg; $\upsilon = 6.04$ Hz; $T = 0.166$ s.]

27. A 3-kg object oscillates on a spring with amplitude of 8 cm. Its maximum acceleration is 3.5 m/s. Find the total energy. [0.42 J]

28. The bob ($m = 0.5$ kg) of a simple pendulum performs SHM with amplitude of 8 cm and a period of 2 s. The motion involves no friction. Calculate maximum values for: a) the speed of the bob b) the kinetic energy of the bob. When the bob has a displacement of 4 cm calculate: c) its velocity, d) its kinetic energy, e) its potential energy. [0.25 m/s; 0.016 J; 0.22 m/s; 0.012 J; 0.004 J]

29. If the period of a pendulum 70 cm long is 1.68 s, what is the value of g at the location of the pendulum? [$g = 9.79$ m/s²]

30. A simple pendulum of length 1 m suspended from the ceiling of an elevator takes 1.8 seconds to complete one oscillation. Find the acceleration of the elevator at its upward motion. [$a =$ 2.4 m/s²]

31. A mass is suspended from a spring and oscillates with a period of 0.980 s. Each complete oscillation results in an amplitude reduction of a factor of 0.96 due to a small velocity dependent frictional effect. Calculate the time it takes for the total energy of the oscillator to decrease to 0.5 of its initial value. [t = 8.33 s]

32. The quality factor of the wire of a musical instrument is 3000. The wire vibrates at a frequency of 300 Hz. Calculate the time in which its amplitude will decrease to half of its initial value. [$t = 2.2$ s]

33. A small pan of mass $m_1 = 0.1$ kg is attached tone end of a spring whose other end is fixed to a rigid support. When a mass $m_2 = 0.9$ kg is placed on the pan, the system performs 240 oscillations per minute and amplitude falls from 2 cm to 1 cm in 60 s. Calculate the force constant, relaxation time and Q-factor. [$k = 625$ N/m; $\tau = 87$ s; $Q = 2175$]

34. The object of the mass $m = 0.1$ kg oscillates on the spring with frequency 0.5 Hz. Its amplitude is known to reduce to half in 2 seconds. Calculate its damping coefficient, spring constant and quality factor. [$\beta = 0.347$ s^{-1}; $k = 0.987$ N/m, $Q = 4.53$]

35. The period of a simple pendulum is 2 s and its amplitude is $5°$. After 20 complete oscillations, its amplitude is reduced to $4°$. Calculate the damping constant and relaxation time.[$\beta = 5.58 \cdot 10^{-3}$ s^{-1}; $\tau = 718$ s]

36. The quality factor of a tuning fork of frequency 512 Hz is 60000. Calculate the time in which its energy drops by factor e. How many oscillations will the tuning fork make in this time? [$N = 9550$; $t = 18.65$ s]

37. A damped mass-spring system oscillated at 200 Hz. The damping coefficient is 0.25 s^{-1}, the amplitude of oscillation id 6 cm and the energy o the oscillating system 9s 60 J. a) What are the amplitudes of oscillation at $t = 2$ s and $t = 4$ s? b) How much energy is dissipated in the first 2-s interval and in the second 2-s interval? [$A(2) = 3.64$ cm; $A(4) = 2.21$ cm; $\Delta W_{0-2} = 37.9$ J; $\Delta W_{2-4} = 24$ J]

38. A damped harmonic system has initial amplitude of 50 cm and damping coefficient 0.05 s^{-1}. a) What will be its amplitude after 35 s? b) How long will take the amplitude to drop to 0.1 % of its original value? c) How long will take the energy of system to be 0.1 % of its original value? [$A = 8.69$ cm; $t_1 = 138.2$; $t_2 = 69.1$ s]

39. For a system executing damped oscillation the time period is 4 s, the logarithmic decrement is 1.6 and $x(0)=0$. The instantaneous displacement of the oscillation at 1 s is 4.5 cm. Write down the expression for displacement of the oscillator. [$x(t)=6.71\cdot e^{-0.4t}\sin(\pi t/2)$(cm)]

40. A compound pendulum with equivalent length of 30 cm executes damped oscillations. In what time will the energy of the oscillation become 20% of the initial energy if the quality factor is 350? [$t=98.6$ s]

41. In a damped oscillatory motion an object oscillates with a frequency of 1 Hz and its amplitude of vibration is halved in 5 s. Find the logarithmic decrement and the equation of oscillations if the object was moves by 5 cm and released. $\delta=0.139$; $x=0.05e^{-0.139t}\cos 2\pi t$]

42. An ocean wave has period 4.1 s and wavelength 108 m. Find the wave number and its angular frequency. [$k=0.582$ m^{-1}; $\omega=1.53$ rad/s]

43. A wave displacement is given by $\xi(x,t)=0.1\sin(0.7x-0.1t)$m. Find the amplitude of the wave, the magnitude of wave vector, the wavelength, the time period, and the wave speed. [$A=0.1$ m; $k=0.1$ m^{-1}, $\lambda=20\pi$ m; $T=20\pi$ s; $v=1$ m/s]

44. Find the amplitude, the wavelength, the period, and the speed of the wave whose displacement is given by $\xi(x,t)=1.3\cos(0.69x+31t)$, where ξ and x are in cm, t is in seconds. In which direction is the wave propagating? [$A=\xi_{\max}=1.3$ cm; $\lambda=9.11$ cm; $T=0.203$ s^{-1}; $v=44.9$ cm/s. The wave is propagating in the negative x-direction]

45. What are the amplitude, the frequency in Hertz, the wavelength, and the speed of a water wave whose displacement is $\xi(x,t)=0.25\sin(0.52x-2.3t)$, where ξ and x are in meters and t is in seconds? [$A=0.25$ m; $\upsilon=0.366$ Hz; $\lambda=12.1$ m; $v=4.42$ m/s]

46. A sound wave with frequency 256 Hz is propagating in air at 343 m/s. How far apart are two points on the wave that differ in phase by $\pi/2$? [$\Delta x=0.335$ m]

47. A wave of frequency 250 Hz has the speed 375 m/s. How far apart are two points 60° out of phase? What is the phase difference between two displacements at a certain point at time 10^{-3} s apart? [0.25 m; $\pi/2$]

48. Two trains move towards each other at a speed of 90 km/h relative to the earth surface. One gives a 520 Hz signal. Find the frequency heard by the observer on the other train (sound velocity 330 m/s). [600 Hz]

49. Two trains move away from each other at a speed of 25 km/h relative to the earth surface. One gives a 520 Hz signal. Find the frequency heard by the observer on the other train (sound velocity 330 m/s). [446.8 Hz]

50. A sound source moves through air toward a stationary observer. The frequency of the sound the observer hears is 20.0% higher than the source frequency. How fast is the source moving? The sound speed in the air is 340 m/s. [56.7 m/s]

REFERENCE LIST

1. Савельев И. В. Курс общей физики. Т. 1. Механика. Молекулярная физика / И. В. Савельев. – М.: Наука, 1988. – 352 с.

2. Савельев И. В. Курс общей физики. Т. 2. Электричество. Колебания и волны. Волновая оптика / И. В. Савельев. – М.: Наука, 1989. – 496 с.

3. Иродов И. Е. Задачи по общей физике / Иродов И. Е. М.: НТЦ «Владис». 1997. – 448 с.

4. Волькенштейн В. С. Сборник задач по общему курсу физики: учебное пособие для вузов / Волькенштейн В. С. – М.: Физматлит, 1990. – 400 с.

5. Чертов А. Г. Задачник по физике / А. Г. Чертов, А. А. Воробьев. – М.: Высшая школа, 1981. – 496 с.

6. Irodov I. E. Problems in General Physics / Igor E. Irodov. – Moscow: Mir Publishers, 1981. – 368 p.

7. Cutnell J. Physics, 10th Ed / John D. Cutnell, Kenneth W. Johnson, David Young, Shane Stadler. – John Wiley & Sons – 992 p.

8. Student Study Guide to accompany Physics, 10th Edition / John D. Cutnell, Kenneth W. Johnson, David Marx – John Wiley & Sons. – 2014. – 428 p.

9. Serway R. Modern Physics / Raymond A. Serway, Clement J. Moses, Curt A. Moyer. – USA. Thomson, Books/Cole, 2005. – 682 p.

10. Hewitt P. G. Conceptual Physics, 10th Ed. / Paul G. Hewitt. – San Francisco: Pearson Education, 2006. – 811 p.

11. College Physics (with Physics Now), 7th Ed. / Serway R. A., Faughn J. S., Vuille C., Bennett C. A. – Brooks Cole, 2005. – 1104 p.

12. Serway R. Essentials College Physics / Raymond A. Serway, Chris Vuille. – Thomson, Books/Cole, 2007. – 885 p.

13. Serway R. A., Faughn J. S. Holt Physics / Serway R. A., Faughn J. S. – McDougal, 2002. – 1006 p.

14. Giancoli D. Physics for scientists and Engineers with Modern Physics / Douglas C. Giancoli. – Pearsons, 2008. – 1322 p.

15. Halliday D. Fundamentals of Physics, 8th Ed. / Halliday D., Resnick R., Walker J. – Hoboken, NY.: John Wiley & Sons, 2007. – 1329 p.

16. Knight R. D. Physics for Scientists and Engineers: A Strategic Approach. 3rd ed. / Randall D. Knight. – Boston: Addison–Wesley. – 2013.

17. Walker J. Fundamentals of physics, 10th Ed / Jearl Walker, David Halliday, Robert Resnick. – Hoboken, NY.: John Wiley & Sons,2014. –1450 p.

18. Giordano N. J. College Physics. Reasoning and Relationships. / Nicolas J. Giordano. – Canada: Brooks/Cole. Cengage Learning, 2010. – 1182 p.

APPENDIX

Table 1 – The Greece Alphabet

Alpha	A	α	Iota	I	ι	Rho	P	ρ
Beta	B	β	Kappa	K	κ	Sigma	Σ	σ
Gamma	Γ	γ	Lambda	Λ	λ	Tau	T	τ
Delta	Δ	δ	Mu	M	μ	Upsilon	Υ	υ
Epsilon	E	ε	Nu	N	ν	Phi	Φ	φ,ϕ
Zeta	Z	ς	Xi	Ξ	ξ	Chi	X	χ
Eta	H	η	Omicron	O	o	Psi	Ψ	ψ
Theta	Θ	ϑ,θ	Pi	Π	π	Omega	Ω	ω

Table 2 – Plane Angle

	Degrees	Minutes	Seconds	Radians	Revolutions
1 degree	1	60	3600	$1.745\cdot10^{-2}$	$2.778\cdot10^{-3}$
1 minute	$1.667\cdot10^{-2}$	1	60	$2.909\cdot10^{-4}$	$4.630\cdot10^{-5}$
1 second	$2.778\cdot10^{-4}$	$1.667\cdot10^{-2}$	1	$4.848\cdot10^{-6}$	$7.716\cdot10^{-7}$
1 radian	57.3	3438	$2.063\cdot10^{5}$	1	0.1592
1 revolution	360	$2.16\cdot10^{4}$	$1.296\cdot10^{6}$	6.283	1

Table 3 – Time

	Years	Days	Hours	Minutes	Seconds
1 year	1	365.25	$8.766\cdot10^{3}$	$5.259\cdot10^{5}$	$3.156\cdot10^{7}$
1 day	$2.738\cdot10^{-3}$	1	24	1440	$8.64\cdot10^{4}$
1 hour	$1.141\cdot10^{-4}$	$4.167\cdot10^{-2}$	1	60	3600
1 minute	$1.9\cdot10^{-6}$	$6.944\cdot10^{-4}$	$1.667\cdot10^{-2}$	1	60
1 second	$3.169\cdot10^{-8}$	$1.157\cdot10^{-5}$	$2.778\cdot10^{-4}$	$1.667\cdot10^{-2}$	1

CONTENT

Навчальне видання

LYUBCHENKO Olena Anatoliivna

MECHANICS

ЛЮБЧЕНКО Олена Анатоліївна

МЕХАНІКА

Навчально-методичний посібник
з курсу «Загальна фізика»
для студентів усіх спеціальностей та усіх форм навчання

Англійською мовою

Роботу до видання рекомендувала доц. С. Д. Гапоченко
В авторській редакції

План 2016 р., поз. 32

Підп. до друку р. Формат 60 х 84 1/16. Папір офсетний.
Riso-друк. Гарнітура Таймс. Ум. друк. арк. Наклад 100 пр.
Зам. № . Ціна договірна.

Видавець і виготовлювач
Видавничий центр НТУ «ХПІ»,
вул. Фрунзе, 21, Харків, 61002
Свідоцтво суб’єкта видавничої справи ДК № 3657 від 24.12.2009 р.